AN

ELEMENTARY TREATISE
ON THE
DIFFERENTIAL
AND
INTEGRAL CALCULUS.

BY

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TRANSLATED FROM THE FRENCH

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TRANSLATOR'S PREFACE.

In presenting to the English student Boucharlat's Treatise on the Differential and Integral Calculus, the Translator feels confident that, if only his own part have been satisfactorily executed, his labours will meet with a favourable reception: the Work itself has already obtained the approbation of the public, for the simplicity and elegance with which a somewhat abstruse subject has been treated.

It is presumed in the original that the reader is already acquainted with the elementary principles of curve lines; and as it will be impossible for him to proceed without at least a slight knowledge of that branch of mathematics, the Translator has been induced, by the advice of his friends, to give a short introductory chapter on that subject. The materials for this he has selected chiefly from Boucharlat's "*Theorie des Courbes*," and though compelled by circumstances to be brief, he trusts that sufficient has been given to enable the student to peruse, without difficulty, the subsequent pages of the Work.

Catharine Hall,
Oct. 30, 1827.

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*** The chapters printed in Italics are such as may be omitted on a first or cursory reading of the subject.

$$AB : V^2 :: ab : 1 \;\ldots\ldots\; (1),$$
$$ABC : V^3 :: abc : 1 \;\ldots\ldots\; (2).$$

If therefore V^2, or the square described on V, be taken as the unit of surface, it appears, from ratio (1), that the rectangle whose sides are A and B, has the same ratio to this square that the algebraical product ab has to unity; ab therefore represents the rectangle AB; and similarly, V^3 being taken for the cubical unit, abc will represent the rectangular parallelopiped whose contiguous sides are a, b, c, and whose volume is abc times V^3.

2. Having thus obtained the means of denoting geometrical magnitudes by algebraical symbols, it follows that the conditions of a geometrical problem may be expressed by those of an algebraical equation; and conversely, the conditions of an algebraical equation may be represented by the relations existing between different geometrical magnitudes.

3. We will give examples of each case; and first of a geometrical problem reduced to an algebraical equation.

Ex. 1. Let it be required to find a mean proportional between two given lines.

Suppose a and b represent the two given lines, and x the line required; we have then, by the question,

$$a : x :: x : b,$$

whence

$$x = \sqrt{ab};$$

and the problem is thus reduced to determining the square root of the product ab, and taking the line, to which that root corresponds.

Fig. 2. Ex. 2. To divide a given straight line AB (fig. 2) into

two parts, so that the rectangle contained by the parts may be equal to a given square.

Let c be a side of the square, a the given line, x one of the parts, and therefore $a-x$ the other: then, by the question, we have

$$x(a-x)=c^2;$$

and the problem reduces itself to determining the roots of the quadratic equation $x(a-x)=c^2$.

These roots are $\frac{1}{2}a+\sqrt{\frac{1}{4}a^2-c^2}$, $\frac{1}{2}a-\sqrt{\frac{1}{4}a^2-c^2}$, and are evidently both positive, since $\frac{1}{4}a^2-c^2$ is less than $\frac{1}{4}a^2$, and therefore $\sqrt{\frac{1}{4}a^2-c^2}$ less than $\sqrt{\frac{1}{4}a^2}$ or $\frac{1}{2}a$.

The side c of the square must be less than $\frac{1}{2}a$, or otherwise $\frac{1}{4}a^2-c^2$ would be negative, when the roots consequently would become imaginary, and the problem impossible.

Ex. 3. On a given line AB (fig. 2) it is required to describe a triangle APB, so that the angle APB at the vertex shall be a right angle. Fig. 2.

Let P be the vertex of the triangle required, PM a perpendicular on the base AB; and assume

$$\mathrm{AM}=a,\ \mathrm{PM}=y,\ \mathrm{AM}=x,\ \text{and therefore } \mathrm{BM}=a-x.$$

Then the triangles APM, BPM being similar, we have

$$\mathrm{AM} : \mathrm{PM} :: \mathrm{PM} : \mathrm{MB},$$

or

$$x : y :: y : a-x,$$

whence

$$x(a-x)=y^2,$$

which is the equation that expresses the relation between

AM and PM, and these lines being known, it is evident that the triangle is known also.

4. The equations that result from these and similar processes must necessarily be homogeneous, or each term of the same number of dimensions. For these equations express the relations existing between geometrical magnitudes, and the species of any one of those magnitudes depends on the dimension of the term which represents it; thus a term a of one dimension represents a straight line; a term a^2 or ab of two dimensions represents a surface, and so on. If therefore an equation be given, which is not homogeneous, it will presume a relation to exist between magnitudes of different kinds, as between a line and a surface; but such a relation is impossible, and the equation therefore cannot be true.

5. We will proceed now to the converse case, and show how the conditions of an algebraical equation may be represented by those of a geometrical problem.

Ex. 4. Having given the equation $x=\sqrt{ab}$; find x. Squaring the equation, we have

$$x^2=ab,$$

and therefore

$$a : x :: x : b.$$

Fig. 2. Let AM, MB, X, (fig. 2) be the lines represented by a, b, x, respectively, then

$$\text{AM} : \text{X} :: \text{X} : \text{MB},$$

and what is required, is to find a mean proportional between the given lines AM, MB; which is thus effected:

Let the lines AM and BM be placed in one straight line AB, so that AM$=a$, BM$=b$; on AB as diameter describe a semicircle APB, and at M erect the perpendicular MP; then, by the property of the circle,

$$AM : MP :: MP : MB,$$

and PM therefore is the mean proportional required, and the symbol representing PM is the value of x.

Ex. 5. Let it be required to find the roots of the equation

$$ax - x^2 = c^2.$$

This may be put under the form

$$x(a - x) = c^2;$$

so that if we take AB (fig. 2) $= a$, AM $= x$, and therefore BM $= a - x$, and also take C $= c$, we shall have Fig. 2.

$$AM . MB = C^2,$$

and the question reduces itself to dividing the line AB into two parts, so that the rectangle contained by the parts shall be equal to the square of a given line C. For this purpose we must describe on AB the semicircle APB, whose centre is G, the middle point of AB; erect GD perpendicular to AB; in GD take GE equal to the given line C, and through E draw PEP′ parallel to AB, and PM perpendicular to AB; then AB will be divided in M, as is required. For by the property of the circle

$$AM . MB = PM^2 = GE^2 = C^2.$$

The proposed equation, being a quadratic, must have two roots, and there must consequently be two points in AB, which answer the conditions; P′M′ being drawn perpendicular to AB, it will be seen that these two points are

M, M′, and the lines corresponding to the two roots or values of x are AM, AM′.

This example is the converse of example (2), and our construction will show us that c must not be greater than $\frac{1}{2}a$; for in that case C would be greater than $\frac{1}{2}$AB, i. e. greater than the radius GD, and when we took GE = C, E would fall above D without the circle, and there would be no point P in the circle corresponding to it.

The process we have been employing is called *constructing* the equation.

6. A problem is said to be determinate or indeterminate, accordingly as it admits of a finite or an indefinite number of solutions.

Thus the example (2) gives us an instance of a determinate problem; for on referring to fig. 2, we see that there are only two points, M and M′, which can satisfy the conditions: example (3), on the contrary, belongs to the class of indeterminate problems; for since the angle in a semicircle is always a right angle, the vertex of the triangle may be any point of the semicircle APB (fig. 2).

A problem is easily recognized as being determinate or indeterminate, from the nature of the equation to which it gives rise: in example (2), for instance, the equation that results is $x(a-x)=c^2$; and this being a quadratic, with only one variable, it can admit of but two solutions. But in example (3) the equation is . . . . $x(a-x)=y^2$, and this being a single equation between two variables x and y, which are not connected in any other way, it belongs to what, in algebra, are termed indeterminate equations.

On curves and their coordinate axes.

7. When a single equation is given between two indeterminate quantities x and y, by assigning different arbitrary values to one of the quantities, as x for instance, an indefinite number of values may be obtained for y. If, for example, we have the equation

$$y=2x+1;$$

making $x=0$, we shall find $y=1$,
$x=1, \ldots\ldots\ldots y=3$,
$x=2, \ldots\ldots\ldots y=5$,
$x=3, \ldots\ldots\ldots y=7$;

and proceeding in this way, it is obvious that we may obtain an unlimited number of corresponding values of x and y, connected always by the equation $y=2x+1$.

8. Let now Ax, Ay (fig. 3) be two straight lines of unlimited length, and at right angles to each other, and let the values of x be represented by lines such as AM measured along Ax from the fixed point A, and the corresponding values of y be similarly represented by lines such as AN, measured from the same point A along Ay; also at the point M draw MP parallel and equal to AN; then AM and MP represent corresponding values of x and y, and if x or AP be supposed to commence from zero, and to pass through every stage of magnitude, the values of y will be represented by an indefinite number of contiguous lines such as MP, all parallel to each other, and the extremities of these lines will, by their union, constitute a continuous line PQ (fig. 3), the form of which will depend on the equation which connects the corresponding values of x and y. Fig. 3.

The line thus traced out is called a *curve;* the equation from which the values of AM, MP are deduced, is called the equation to the curve, and the curve itself is said to be the *locus* of the equation.

9. It will be seen from this that every indeterminate equation must have some curve corresponding to it, and conversely every curve must be the locus of some equation: it may either be required from the equation to determine the figure and properties of the curve; or from given properties of the curve to determine its equation; before entering on this subject, however, we must make some further remarks on the lines Ax, Ay, and AM, MP measured along those lines.

10. The lines AM, AN, or AM, MP, when spoken of together, are called the coordinates of the point P; to distinguish them from each other, AM is also called the abscissa, and MP the ordinate of that point; the indefinite lines Ax, Ay we denominate the axes, and A the origin of the coordinates.

The axes may be inclined at any angle to each other, but they are generally drawn at right angles, and the coordinates are then termed rectangular.

11. Hitherto the coordinates have been supposed to be measured only towards x and y; but if the axes be taken to extend indefinitely from A in the directions of x, x', and y, y', respectively, we may suppose the coordinates to be measured also towards x' and y'.

Now if a line measured in one direction be supposed positive, a line measured in the *opposite* direction must be supposed negative; and if therefore the values of x measured to the right of A be taken as positive, those to the left of A, or towards x', must be considered negative, and similarly for the values of y.

Hence for a point P in the plane yAx we have $+x$, $+y$,
P' . . . . yAx' . . . . . . $-x$, $+y$,
P'' . . . . $y'Ax'$ . . . . . $-x$, $-y$,
P''' . . . . $y'Ax$ . . . . . $+x$, $-y$.

12. The position of any point in a plane may be determined if its coordinates, measured from a given point, are known; for let A (fig. 3) be the origin of the coordinates, and a, b, the abscissa and ordinate of the point: if, then, along the axis Ax we take $AM=a$, and draw MP, an indefinite line, parallel to Ay, the point required must be in the line MP, for that line comprises all points which have a for their abscissa; similarly if we take $AN=b$, and draw NP parallel to Ax, the point must be in that line, and it must therefore be in P, the intersection of the lines MP, NP. When the coordinates of the point are known, they are generally represented by a, b; wh[illegible] unknown or variable, by x, y. Fig. 3.

13. Another method of determining the position of a point in a plane is also used, in which, instead of both axes being fixed as before, one of the axes, as xx' (fig. 4) and the origin A are supposed to be fixed, whilst the other axis yy' is inclined to xx' at a variable angle. Fig. 4.

On this supposition, let P be any point, and make the axis yy' pass through it; then it is clear that P may be determined if we know the angle $x'Ay$ at which the axis Ay is inclined to Ax, and also the distance AP measured along Ay.

The origin A is called the *pole*, AP the *radius vector*, and the equation between the angle $x'Ay$, and the distance AP, is called the *polar* equation. AP is represented generally by u or r, and the angle $x'Ay$ by θ.

14. An equation between rectangular coordinates is readily transformed into one between polar, and the con-

MP, of any point P by x, y, and draw BN perpendicular to PM.

Then

$$\begin{aligned} y = \text{PM} = \text{PN} + \text{MN} &= \text{PN} + \text{AB} \\ &= \text{BN}.\tan\text{PBN} + \text{AB} \\ &= \text{AM}.\tan\text{ACB} + \text{AB} \\ &= \quad x.\tan\text{ACB} + \text{AB}. \end{aligned}$$

Now the angle ACB, and consequently its tangent, is the same for all points in the line CBP, as is also AB, and we will therefore represent these constant quantities by a, b, respectively, when we shall have

$$y = ax + b \quad . \ . \ . \ . \quad (3):$$

which is the equation that exists between the coordinates of any point P in the line CBP, and is therefore the equation of that line.

This equation is of the first degree, and it appears therefore that the equation to a straight line is of the first degree.

17. It may be shown conversely that every equation of the first degree is the equation to some straight line. For, by inspecting the equation (3), it will be seen that a, the constant coefficient of x, expresses the tangent of the angle which the line CBP makes with the axis of x, whilst b is the ordinate of the point in which that line cuts the axis of y. If, therefore,

$$y = \text{A}x + \text{B},$$

which is a general simple equation, be the equation proposed, by taking AB = B (fig. 7), and drawing the line Fig. 7. BC inclined to Ax at an angle whose tangent is A, this line produced will be the one whose equation is the given equation

$$y = Ax + B.$$

If $B = 0$, the line passes through the origin A.

If B be negative, the line cuts the axis of y in a point below A, as B′C′P′ (fig. 8). Fig. 8.

If A be positive, the angle PCx is less than a right-angle; but if A be negative, that angle becomes obtuse, and the line assumes the direction BCP (fig. 8).

If $A = 0$, then $y = B$ is constant, and the line is parallel to Ax.

Ex. 1. Given the simple equation $y = 2x + 3$, required the line of which it is the equation.

The equation being

$$y = 2x + 3,$$

w[illegible] $x = 0$, we have $y = 3$, and therefore taking $AB = 3$ (fig. 7), B will be a point in the line: Fig. 7.
when $y = 0$, we have similarly $x = -\frac{3}{2}$, and therefore taking AC in a negative direction and $= \frac{3}{2}$, C will be another point; consequently joining BC, this line produced will be the one required.

Ex. 2. $y = 3x - 6$: required the line of which this is the equation.

Since

$$y = 3x - 6,$$

when $x = 0$, we have $y = -6$,
when $y = 0$, we have $x = 2$;
and therefore taking AB′ (fig. 8) in a negative direction, Fig. 8.
and $= 6$, and taking AC′ in a positive direction, and $= 2$, the line B′C′ produced will be the one whose equation is $y = 3x - 6$.

Ex. 3. $2y=8-5x$: required the line of which it is the equation.

From this we deduce

$$y=4-\frac{5x}{2},$$

and making $x=0$, we have $y=4$,

making $y=0$, we have $x=\frac{8}{5}$;

Fig. 8. whence taking $AB=4$ (fig. 8), and $AC=\frac{8}{5}$, the line CBP, passing through the points B, C, will be the line required.

Problems relating to straight lines.

18. Prob. 1. To find the equation to a straight line passing through a given point.

The point being given, its coordinates are known; let them be α, β, and let the equation to the straight line be

$$y=ax+b \quad . \; . \; . \; . \quad (1).$$

At the point in question we have $x=\alpha$, $y=\beta$, and therefore

$$\beta=a\alpha+b \quad . \; . \; . \; . \quad (2);$$

whence, subtracting equations (1) and (2), one from the other, we obtain

$$y-\beta=a(x-\alpha),$$

which is the equation to the line.

The constant a in this equation cannot be determined from the conditions given, so that the equation is indeterminate; and so it ought to be; for it is easily seen that through the same point an infinite number of lines may be drawn.

Prob. 2. Required the equation to a straight line passing through two given points.

Let α, β; α', β', be the coordinates corresponding to the two points; and let the equation required be

$$y = ax + b \quad . \quad . \quad . \quad . \quad (1);$$

then for the given points we have

$$\beta = a\alpha + b \quad . \quad . \quad . \quad . \quad (2),$$
$$\beta' = a\alpha' + b \quad . \quad . \quad . \quad . \quad (3);$$

subtracting equation (3) from equation (2), we have

$$\beta - \beta' = a(\alpha - \alpha'),$$

and therefore

$$a = \frac{\beta - \beta'}{\alpha - \alpha'}:$$

subtracting equation (2) from equation (1), there remains

$$y - \beta = a(x - \alpha),$$

and substituting the value of a, we have

$$y - \beta = \frac{\beta - \beta'}{\alpha - \alpha'}(x - \alpha)$$

for the equation required.

This equation is determinate; and so it manifestly should be, since only one straight line can pass through the same two points.

If there be three points taken whose coordinates are

α, β; α', β'; α'', β'', respectively, the equation to the straight line passing through the two first points is

$$y-\beta=\frac{\beta-\beta'}{\alpha-\alpha'}(x-\alpha);$$

the equation to the line passing through the first and third points is

$$y-\beta=\frac{\beta-\beta''}{\alpha-\alpha''}(x-\alpha);$$

and therefore, that the two equations may be the same, or the same line pass through all the three points, we must have

$$\frac{\beta-\beta'}{\alpha-\alpha'}=\frac{\beta-\beta''}{\alpha-\alpha''}.$$

The equation to the straight line passing through two given points may readily be obtained by geometrical considerations: for let (fig. 9) E, G, be the given points, P any other point in the line passing through E, G, and draw the ordinates EF, GH, PM; and ELN perpendicular to PN.

Fig. 9.

Then from the similar triangles EPN, ELG, we have

$$\text{PN} : \text{LG} :: \text{EN} : \text{EL};$$

whence

$$\text{PN}=\frac{\text{LG}}{\text{EL}}\cdot\text{EN},$$

or

$$\text{PM}-\text{EF}=\frac{\text{GH}-\text{EF}}{\text{AH}-\text{AF}}(\text{AM}-\text{AF}),$$

and if the coordinates AF, EF; AH, GH; AM, MP, be represented by α, β; α', β; x, y, respectively, we shall have, by substitution,

$$y-\beta=\frac{\beta'-\beta}{\alpha'-\alpha}(x-\alpha),$$

the same equation with the one deduced before.

Prob. 3. Given the equations to two lines, required their point of intersection.

Let the equations to the lines be

$$y=ax+b,\ y'=a'x'+b':$$

then at the point of intersection, the coordinates being the same for both lines, we must have

$$y=y',\ x=x',$$

and therefore

$$ax+b=a'x+b'$$

$$x(a-a')=b'-b$$

$$x=\frac{b'-b}{a-a'};$$

whence

$$\begin{aligned} y&=ax+b\\ &=\frac{ab'-ab}{a-a'}+b\\ &=\frac{ab'-a'b}{a-a'}; \end{aligned}$$

and these values of x and y give the coordinates at the point of intersection.

Prob. 4. Required the angle formed by the intersection of two given lines.

Fig. 10. Let CBP, C′B′P′ (fig. 10) be the two lines, and their equations

$$y = ax + b,\ y' = a'x' + b';$$

then

$$\tan ACB = a,\ \tan AC'B' = a':$$

If, now, O be the point of intersection and ON be drawn parallel to Ax, we shall have

$$\begin{aligned}\text{angle } POP' &= PON - P'ON,\\ &= ACB - AC'B',\end{aligned}$$

and therefore

$$\begin{aligned}\tan POP' &= \tan(ACB - AC'B')\\ &= \frac{\tan ACB - \tan AC'B'}{1 + \tan ACB . \tan AC'B'}\\ &= \frac{a - a'}{1 + aa'}.\end{aligned}$$

Prob. 5. A straight line being given, to find the equation to the line perpendicular to it at a given point.

Fig. 11. Let CBP (fig. 11) be the given line, C′PB′ the line perpendicular to it at the point P, whose coordinates are α, β; and let the equations to the lines be

$$y = ax + b,\ y' = a'x' + b';$$

then

$$\tan PCx = a,\ \tan PC'x = a'.$$

But CPC′ being a right angle, we have

$$\tan PC'x = -\tan PC'C = -\cot PCC' = -\frac{1}{\tan PCC'},$$

and therefore

$$a' = -\frac{1}{a}.$$

Substituting this value in the equation to C′PB′, we have

$$y' = -\frac{1}{a}x' + b' \quad . \; . \; . \quad (1),$$

and since P, whose coordinates are α, β, is a point in the line, we have for that point,

$$\beta = -\frac{1}{a}\alpha + b' \quad . \; . \; . \quad (2),$$

whence, subtracting equation (2) from equation (1), we obtain

$$y' - \beta = -\frac{1}{a}(x' - \alpha)$$

for the equation to the line C′PB′.

Prob. 6. Find the length of the perpendicular let fall from a given point on a given line.

Let CBP (fig. 11) be the straight line, M the point from which the perpendicular MP is let fall; α, β the coordinates of that point, and the equation to the line CBP Fig. 11

$$y = ax + b.$$

Then in the triangle PMO we have

$$\text{angle PMO} = \frac{\pi}{2} - \text{POM} = \frac{\pi}{2} - \text{CON} = \text{ACB};$$

but the equation $y = ax + b$ gives us tan ACB $= a$, and consequently

$$\tan \text{PMO} = a,$$

whence

$$\cos \mathrm{PMO} = \frac{1}{\sqrt{1+a^2}}\;*.$$

Also

$$\mathrm{MO} = \mathrm{MN} - \mathrm{ON},$$

and O being a point in the line which corresponds to the abscissa $\mathrm{AN} = \alpha$, we have

$$\mathrm{ON} = a\alpha + b,$$

whence

$$\begin{aligned} \mathrm{MO} &= \mathrm{MN} - \mathrm{ON} \\ &= \beta - (a\alpha + b), \end{aligned}$$

and therefore

$$\begin{aligned} \mathrm{MP} &= \mathrm{MO} \,.\, \cos \mathrm{PMO} \\ &= \frac{\beta - a\alpha - b}{\sqrt{1+a^2}}. \end{aligned}$$

---

* Let $\tan .\, \theta = a$;

$$\begin{aligned} \text{then } \cos^2\theta &= \frac{\sin^2\theta}{\tan^2\theta} \\ &= \frac{1-\cos^2\theta}{a^2}, \end{aligned}$$

whence

$$\begin{aligned} a^2\cos^2\theta + \cos^2\theta &= 1, \\ \cos^2\theta &= \frac{1}{1+a^2} \\ \cos\theta &= \frac{1}{\sqrt{1+a^2}}. \end{aligned}$$

Also

$$\sin\theta = \frac{a}{\sqrt{1+a^2}}.$$

Equations to the conic sections.

19. *To find the equation to the circle.*

Let α and β be the coordinates AB, BC (fig. 12) of the centre C of a circle PP′P″ described with a radius $CP=r$. Take any point P in the circle, draw the ordinate PN perpendicular to Ax, and CM perpendicular to PN, and let the coordinates AN, PN be represented by x, y: then from the right-angled triangle CMP we have Fig. 12.

$$\begin{aligned} CP^2 &= CM^2+PM^2 \\ &= (AN-AB)^2+(PN-BC)^2, \end{aligned}$$

or

$$r^2=(x-\alpha)^2+(y-\beta)^2,$$

which is the general equation to the circle.

If the origin A be a point in the circumference of the circle, and the axis of x be the diameter, then we have

$$\alpha=r,\ \beta=0,$$

and the equation becomes

$$r^2=(x-r)^2+y^2,$$

whence

$$\begin{aligned} y^2 &= r^2-(x-r)^2 \\ &= 2rx-x^2 \\ y &= \pm\sqrt{2rx-x^2}. \end{aligned}$$

If the centre C of the circle be made the origin, then

$$\alpha=0,\ \beta=0,$$

and the equation, therefore, is reduced to

$$r^2=x^2+y^2,$$

whence

$$y=\pm\sqrt{r^2-x^2}.$$

20. The general equation

$$r^2=(x-\alpha)^2+(y-\beta)^2$$

being solved with regard to y, gives us

$$y=\beta\pm\sqrt{r^2-(x-\alpha)^2};$$

and with regard to x, gives

$$x=\alpha\pm\sqrt{r^2-(y-\beta)^2}.$$

It appears, therefore, from these expressions, first, that for every value of x we have two values of y, and secondly that for every value of y we have two of x.

Now taking any abscissa $AN=x$, the ordinate NP cuts the circle in two points P, P'', and therefore the two values of y corresponding to the single one AN of x are

$$PN=\beta+\sqrt{r^2-(x-\alpha)^2},$$
$$P''N=\beta-\sqrt{r^2-(x-\alpha^2)}.$$

Again, drawing PP' parallel to Ax, we have $PN=P'N'$, and the value of y is the same for the two points P, P', which have different abscissæ AN, AN', so that for one value PN of y we have two values of x

$$AN=\alpha+\sqrt{r^2-(y-\beta)^2},$$
$$AN'=\alpha-\sqrt{r^2-(y-\beta)^2}.$$

21. *To find the equation to the parabola.*

Fig. 13. Let DKR (fig. 13) be a fixed straight line of indefinite extent, S a given point, SP another straight line of indefinite extent revolving about S. If then in all positions of the line SP, the point P be so taken that, PM being drawn perpendicular to DR, PM shall be equal to

SP, the locus of P will be a curve AP called the *parabola.*

Through S draw SK perpendicular to DR, and bisect SK in A; then from our definition A will be a point in the curve.

Let the line AS produced be taken for the axis of x, and A the origin; draw the ordinate PN, and let

$$AS=a,\ AN=x,\ PN=y.$$

Now by the property of the curve, we have SP=PM; and

$$SP^2=SN^2+PN^2=(AN-AS)^2+PN^2$$
$$PM^2=KN^2=(AK+AN)^2=(AS+AN)^2;$$

whence

$$(AN-AS)^2+PN^2=(AS+AN)^2,$$

or

$$(x-a)^2+y^2=(a+x)^2$$

and therefore

$$\begin{aligned} y^2 &= (a+x)^2-(x-a) \\ &= 4ax \end{aligned}$$

which is the equation required.

Let BL be the ordinate through S, then, since AS=a,

$$y^2=4a^2,$$
$$y=2a,$$

and therefore

$$SL=2AS.$$

DR is called the director, A the vertex, S the focus, and BL the latus rectum of the parabola.

Fig. 14. 22. Let S and H (fig. 14) be two fixed points, and suppose that the indefinite straight lines SP, HP revolve about S, H, intersecting in P so that SP+HP shall always be equal to a given quantity: then the point P traces out a curve called the ellipse.

Join S, H, and produce the line SH to the curve in A, M; bisect SH in C, and through C draw BCD perpendicular to SH, and cutting the curve in the points B, D: then C is called the centre of the ellipse; S, H, the foci; A, M, the vertices; AM the major, and BD the minor axes.

23. The major axis AM is equal to the sum of the focal distances SP + HP.

For, by the property of the curve, we have

$$SP + HP = SA + HA,$$
$$SP + HP = SM + HM,$$

and therefore adding these, there results

$$2(SP + HP) = AM + AM = 2AM,$$

and consequently

$$SP + HP = AM$$

The triangles SBC, HBC, are manifestly equal and similar, and therefore

$$SB = HB;$$

but

$$SB + HB = AM,$$

whence

$$2\,SB = 2\,AC,$$

and

$$SB = AC = \tfrac{1}{2} \textit{ major axis.}$$

The ratio of SC to AC measures what is called the *eccentricity* of the ellipse, so that the farther S is taken from C, AC remaining the same, the greater is the eccentricity: a circle may be considered as an ellipse in which S, H, meet in the centre C, or in which the eccentricity is 0.

24. If the lines SP, HP, be so taken that the excess of HP over SP, or (HP—SP), be constantly the same, then P traces out a curve AP (fig. 15) called the hyperbola. Fig. 15.

25. *To find the equation to the ellipse.* Let C the centre of the ellipse ABM (fig. 14) be taken for the origin of the coordinates, and the major axis for the axis of x: P being any point in the curve, draw the ordinate PN, and suppose Fig. 14.

$$AC=a,\ BC=b,\ SC=\sqrt{a^2-b^2}=c,\ CN=x,\ PN=y:$$

then $SP=\sqrt{SN^2+PN^2}=\sqrt{\overline{c+x}^2+y^2}$,

$$HP=\sqrt{HN^2+PN^2}=\sqrt{\overline{c-x}^2+y^2}.$$

But $SP+HP=2a$ by property of the ellipse, and therefore

$$2a=\sqrt{\overline{c+x}^2+y^2}+\sqrt{\overline{c-x}^2+y^2},$$

whence

$$2a-\sqrt{\overline{c-x}^2+y^2}=\sqrt{\overline{c+x}^2+y^2};$$

and squaring, and employing the common processes of Algebra, we have

$$4a^2-4a\sqrt{(c-x)^2+y^2}+(c-x)^2+y^2=(c+x)^2+y^2,$$

$$4a^2-4a\sqrt{(c-x)^2+y^2}=(c+x)^2-(c-x)^2=4cx,$$

$$a^2-a\sqrt{(c-x)^2+y^2}=cx,$$

whence, by transposition and change of sign,

$$a\sqrt{(c-x)^2+y^2}=a^2-cx;$$

and squaring again, we have

$$a^2(c-x)^2+a^2y^2=a^4-2a^2cx+c^2x^2,$$

or

$$\begin{aligned}a^2y^2&=a^4-2a^2cx+c^2x^2-a^2(c^2-2cx+x^2)\\&=a^4+c^2x^2-a^2c^2-a^2x^2\\&=a^4+x^2(a^2-b^2)-a^2(a^2-b^2)-a^2x^2\\&=-b^2x^2+a^2b^2\\&=b^2(a^2-x^2),\end{aligned}$$

and therefore

$$y^2=\frac{b^2}{a^2}(a^2-x^2),$$

which is the equation to the ellipse.

If the origin be transferred to the point A, whose coordinates, reckoned from origin C, are $-a$, 0, then, by the rule given (art. 15) for changing the origin, we have, for the coordinates AN, PN, measured from A, the equation

$$\begin{aligned}y^2&=\frac{b^2}{a^2}(a^2-(x-a)^2)\\&=\frac{b^2}{a^2}(2ax-x^2).\end{aligned}$$

26. The equation to the hyperbola, deduced in a similar manner, is

for *origin* C, $y^2=\dfrac{b^2}{a^2}(x^2-a^2)$,

$$\text{for } \textit{origin}\ A,\quad y^2=\frac{b^2}{a^2}(2ax+x^2).$$

27. Let SL (fig. 14) be the ordinate through S the focus, then the value of x corresponding to this ordinate is CS or $\sqrt{a^2-b^2}$, and substituting this value in the equation to the ellipse Fig. 14.

$$y^2=\frac{b^2}{a^2}(a^2-x^2),$$

we have

$$\begin{aligned} y^2&=\frac{b^2}{a^2}(a^2-\overline{a^2-b^2})\\ &=\frac{b^2}{a^2}.b^2\\ &=\frac{b^4}{a^2}, \end{aligned}$$

whence

$$y=\frac{b^2}{a},$$

which gives us the value of SL.

The line LL′, drawn through the focus at right angles to the major axis, is called the Latus Rectum of the ellipse, and the value of the Latus Rectum therefore is $\frac{2b^2}{a}$, which shows that it is a mean proportional between the major and minor axes.

28. Let APBM (fig. 16) be an ellipse with AM, BE for its major and minor axes: on AM as diameter describe the circle ADM, and draw the ordinate QPN: then A being considered the origin, Fig. 16.

the equation to the circle is $y^2 = 2ax - x^2 = QN^2$,

the equation to the ellipse is $y^2 = \frac{b^2}{a^2}(2ax - x^2) = PN^2$,

and we have therefore

$$PN^2 : QN^2 :: \frac{b^2}{a^2} : 1 :: b^2 : a^2,$$

or

$$PN : QN :: b : a,$$

which is a constant ratio.

This property gives us the following easy mode of describing an ellipse: take a circle ADM (fig. 16) on diameter ACM, at the centre C erect CD at right angles to AM, and take any fixed point B in it: then if we draw ordinates such as QN, and take always in QN the point P, so that Fig. 16.

$$PN : QN :: BC : AC,$$

the locus of P will be an ellipse with AM and BE for its major and minor axes.

It will be evident from this that an indefinite number of ellipses may be described on the same line ACM as the major axis; for an indefinite number of points such as B may be taken in the line CB, and for each point we shall have an ellipse. The position of the focus S will be determined in each case by taking $CS = \sqrt{a^2 - b^2}$, where a, b, are the semi-major and minor axes.

29. The four curves now described, the circle, the ellipse, the parabola, and the hyperbola, are called the conic sections, from the circumstance that if any right-angled cone be cut by a plane, perpendicular to the plane

of the generating triangle, the section of the cone will be one or other of these curves.

Let ABD (fig. 17) be the right-angled cone, BCD its circular base: if then the cutting plane be parallel to the base, the section will evidently be a circle. Fig. 17.

If the plane cut both slant sides of the cone, the section will be an ellipse.

If the plane be parallel to one of the slant sides, the section will be a parabola.

If, lastly, the plane have to be produced backwards to meet one of the slant sides, the section will be an hyperbola. Fig. 18.

These properties are thus proved:

30. Let ABC (fig. 17) be a right cone generated by the revolution of the plane isosceles triangle ABC about the axis AD, and suppose this cone to be cut by a plane NO perpendicular to the plane of the triangle ABC: let NO be the common section of the planes, NPOQ the section of the cone, PMQ any ordinate at right angles to NO; EF a section of the cone made by a plane passing through PQ perpendicular to the triangle ABC, and parallel to the base BC, when of course EF will be a circle, and also the line PQ, being in the plane NO, which is perpendicular to ABC, will itself be perpendicular to ABC, and therefore at right angles to the line EF in that plane. Fig. 17.

Now let

$$\angle \text{BAC} = \alpha, \ \angle \text{AON} = \theta, \text{ and therefore } \angle \text{ANO}$$
$$= \pi - (\alpha + \theta), \ \text{AO} = c, \ \text{NM} = x, \ \text{PM} = y:$$

then, by property of the circle, we have

$$\text{PM}^2 = \text{EM} \times \text{MF},$$

or

$$y^2 = \mathrm{EM} \times \mathrm{MF} \quad . \; . \; . \; . \quad (1);$$

and, by the property of triangles,

$$\mathrm{EM} = \mathrm{MN} . \frac{\sin \mathrm{ENM}}{\sin \mathrm{NEM}} = \mathrm{MN} . \frac{\sin \mathrm{ANO}}{\sin \mathrm{AEF}}$$

$$= x . \frac{\sin(\pi - \overline{\alpha + \theta})}{\sin\left(\frac{\pi}{2} - \frac{\alpha}{2}\right)} = x . \frac{\sin(\alpha + \theta)}{\cos\frac{\alpha}{2}}:$$

$$\mathrm{MF} = \mathrm{MO} . \frac{\sin \mathrm{MOF}}{\sin \mathrm{MFO}} = (\mathrm{NO} - \mathrm{NM}) . \frac{\sin \mathrm{AON}}{\sin \mathrm{AFE}}$$

$$= (\mathrm{NO} - \mathrm{NM}) . \frac{\sin \theta}{\cos\frac{\alpha}{2}}:$$

$$\mathrm{NO} = \mathrm{AO} . \frac{\sin \mathrm{NAO}}{\sin \mathrm{ANO}} = c . \frac{\sin \alpha}{\sin(\alpha + \theta)}.$$

Substituting these values in the equation (1) we obtain

$$y^2 = x . \frac{\sin \theta . \sin(\alpha + \theta)}{\cos^2 \frac{\alpha}{2}} \left(\frac{c . \sin \alpha}{\sin(\alpha + \theta)} - x \right)$$

$$= x . \frac{\sin \theta}{\cos^2 \frac{\alpha}{2}} \left(c . \sin \alpha - x . \sin(\alpha + \theta) \right) \quad . \; . \quad (2),$$

which is the equation to the section of the cone.

This equation will admit of four cases, according to the position of NO.

1°. Let NO meet both slant sides of the ellipse; then $(\alpha + \theta)$ will be less than π, $\sin(\alpha + \theta)$ will be positive, and the equation (2) may be put under the form

$$y^2 = \frac{\sin(\alpha+\theta)\sin\theta}{\cos^2\frac{\alpha}{2}}\left(\frac{c.\sin\alpha}{\sin(\alpha+\theta)}x - x^2\right) \;.\;.\;.\;.\;.\; (3);$$

but the equation to an ellipse whose semi-major and minor axes are a and b, is $y^2 = \frac{b^2}{a^2}(2ax - x^2)$, and comparing equation (3) with this, we see that equation (3) belongs to an ellipse, in which

$$2a = \frac{c.\sin\alpha}{\sin(\alpha+\theta)},\quad a = \frac{c}{2}\cdot\frac{\sin\alpha}{\sin(\alpha+\theta)};$$

$$\frac{b^2}{a^2} = \frac{\sin(\alpha+\theta)\sin\theta}{\cos^2\frac{\alpha}{2}},$$

and therefore

$$b^2 = \frac{c^2}{4}\cdot\frac{\sin^2\alpha}{\sin^2(\alpha+\theta)}\cdot\frac{\sin(\alpha+\theta)\sin\theta}{\cos^2\frac{\alpha}{2}}$$

$$= \frac{c^2}{4}\cdot\frac{\sin^2\alpha}{\cos^2\frac{\alpha}{2}}\cdot\frac{\sin\theta}{\sin(\alpha+\theta)},$$

whence

$$b = \frac{c}{2}\cdot\frac{\sin\alpha}{\cos\frac{\alpha}{2}}\sqrt{\frac{\sin\theta}{\sin(\alpha+\theta)}}$$

$$= c.\sin\frac{\alpha}{2}\sqrt{\frac{\sin\theta}{\sin(\alpha+\theta)}}.$$

The section, therefore, is an ellipse with axes thus determined.

2°. Let NO be parallel to one of the slant sides as AB, then $(\alpha+\theta)=\pi$, $\sin(\alpha+\theta)=0$, the values of the axes become infinite, and the equation (2) becomes

$$y^2=\frac{c.\sin\theta.\sin\alpha}{\cos^2\frac{\alpha}{2}}x$$

$$=\frac{c.\sin(\pi-\alpha)\sin\alpha}{\cos^2\frac{\alpha}{2}}x$$

$$=\frac{c.\sin^2\alpha}{\cos^2\frac{\alpha}{2}}x$$

$$=\frac{4c.\sin^2\frac{\alpha}{2}.\cos^2\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2}}x$$

$$=4c.\sin^2\frac{\alpha}{2}.x,$$

which is the equation to a parabola, in which, if a be the Latus Rectum,

$$a=4c.\sin^2\frac{\alpha}{2}.$$

The section therefore is a parabola of that description.

3°. If NO have to be produced backwards to meet BA
Fig. 18. also produced as in (fig. 18), then $\alpha+\theta$ becomes greater than π, and $\sin(\alpha+\theta)$ becomes negative, so that the equation (2) will agree with the equation to the hyperbola

$$y^2=\frac{b^2}{a^2}(2ax+x^2),$$

the axes being determined as in the ellipse.

4°. If the section be parallel to the base of the cone, then

$$\theta=\frac{\pi}{2}-\frac{\alpha}{2},\quad \alpha+\theta=\frac{\alpha}{2}+\frac{\pi}{2},$$

and

$$\sin(\alpha+\theta)=\sin\left(\frac{\alpha}{2}+\frac{\pi}{2}\right)=\cos\frac{\alpha}{2},$$

$$\sin\theta\quad=\sin\left(\frac{\pi}{2}-\frac{\alpha}{2}\right)=\cos\frac{\alpha}{2};$$

so that equation (2) becomes

$$y^2=\frac{\cos^2\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2}}\left(\frac{c.\sin\alpha}{\cos\frac{\alpha}{2}}x-x^2\right)$$

$$=2c.\sin\frac{\alpha}{2}x-x^2;$$

which is the equation to a circle whose radius is $c.\sin\frac{\alpha}{2}$. This is, in fact, a particular instance of case 1, in which θ is taken equal to $\frac{\pi}{2}-\frac{\alpha}{2}$; on this supposition, the values of a and b become

$$a=\frac{c}{2}\cdot\frac{\sin\alpha}{\cos\frac{\alpha}{2}}=c.\sin\frac{\alpha}{2},$$

$$b=c\sin\frac{\alpha}{2}\sqrt{\frac{\cos\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}}=c.\sin\frac{\alpha}{2},$$

which shows that the axes are equal, and the ellipse consequently becomes a circle.

31. The equations which we have deduced to the conic sections are all of the quadratic form, and it may be shown generally that every quadratic equation belongs to some one of those sections.

The form of a general quadratic equation between two variables x and y is

$$Ay^2+Bxy+Cx^2+Dy+Ex+F=0 \quad . \ . \ . \ (1);$$

and this being an indeterminate equation, it must (art. 9) have some curve corresponding to it whose coordinates are x and y.

Let the origin of these coordinates be transferred (art. 15) to some other point, the coordinates of which are a and b, by assuming

$$x=x'+a, \quad y=y'+b;$$

then the equation (1) will become

$$\left.\begin{array}{l} A(y'^2+2by'+b^2)+B(x'y'+x'b+y'a+ab) \\ +C(x'^2+2ax'+a^2)+D(y'+b)+E(x'+a)+F \end{array}\right\} = 0,$$

in which the coefficients of x' and y' are $Bb+2aC+E$, and $Ba+2Ab+D$ respectively; and a and b being two indeterminate quantities, they may be so assumed that these coefficients shall be each 0, for which purpose we must have the equations

$$Bb+2aC+E=0$$
$$Ba+2bA+D=0,$$

whence we shall deduce, for the required values of a and b,

$$a=\frac{\text{BD}-2\text{AE}}{4\text{AC}-\text{B}^2},$$

$$b=\frac{\text{BE}-2\text{CD}}{4\text{AC}-\text{B}^2}.$$

If, therefore, in the curve corresponding to equation (1), the origin be changed to a point, whose coordinates are a and b thus determined, we shall obtain an equation which does not involve the simple powers of x and y, and which, consequently, may be assumed to be of the form

$$\text{A}y^2+\text{B}xy+\text{C}x^2+\text{D}=0\ \ .\ .\ .\ (2).$$

Let now the equation (2) be transferred to a system of coordinate axes inclined at an angle θ to the former, by assuming (art. 15)

$$x=x'.\cos\theta-y'.\sin\theta,$$
$$y=y'.\cos\theta+x'.\sin\theta;$$

when we shall have

$$\begin{aligned}&\text{A}(y'^2\cos^2\theta+2x'y'\sin\theta.\cos\theta+x'^2\sin^2\theta)\\&+\text{B}[x'y'(\cos^2\theta-\sin^2\theta)-(y'^2-x'^2)\sin\theta\cos\theta]\\&+\text{C}(y'^2.\cos^2\theta+2x'y'\sin\theta\cos\theta+x'^2\sin^2\theta)+\text{D}=0,\end{aligned}$$

in which the coefficient of $x'y'$ is

$$\text{A}\sin 2\theta+\text{B}\cos 2\theta+\text{C}\sin 2\theta,$$

and if we suppose this to be 0, we shall have, dividing by $\cos 2\theta$,

$$\text{A}\tan 2\theta+\text{C}\tan 2\theta+\text{B}=0,$$

whence

$$\tan 2\theta = -\frac{B}{A+C};$$

and if, therefore, θ be assumed so as to answer this condition, the resulting equation to the curve will involve only the second powers of x and y, and constants, and will consequently be of the forms

$$Py^2 + Qx^2 = R,$$

or

$$y^2 = \frac{R}{P} - \frac{Q}{P}x^2,$$

and making

$$\frac{R}{P} = m, \quad \frac{Q}{P} = n,$$

we shall have

$$y^2 = m - nx^2,$$

$$= n\left(\frac{m}{n} - x^2\right) \ldots (3),$$

which is the equation to the ellipse or hyperbola as n is positive or negative. This is the equation to the ellipse or hyperbola, when the centre C is considered as the origin of the coordinates, and it will therefore be seen that a and b are the coordinates of that centre, measured from the origin which corresponds to our equation (1). If now the coefficients A, B, C, in that equation be such that $4AC - B^2 = 0$, the values of a, b, as there deduced, will become infinite, and the axes consequently will be infinite, in which case the curve cannot be either an ellipse or hyperbola. Under these circumstances we must get quit of the terms involving x^2, xy, y, and the constants, when our equation will be reduced to the form

$$y^2 = mx,$$

which is the equation to the parabola.

It appears from all this, that the general equation (1) may, by changing the position and direction of the axes, be in all cases reduced to the form of the equation belonging to one or other of the conic sections; and since this change of the axes cannot affect the nature of the curve, the curve corresponding to equation (1) must always be one or other of those sections.

32. When $B^2-4AC=0$, the axes of the conic section are found to be infinite, and the ellipse corresponding to the equation in that case might be considered as one with an infinite axis: it may be shown that such an ellipse is in fact a parabola. For taking A (fig. 14) for the origin, the equation to the ellipse is Fig. 14.

$$y^2=\frac{b^2}{a^2}(2ax-x^2),$$

and if a be supposed to be infinite, x^2 vanishes in comparison with $2ax$, and the equation becomes

$$y^2=\frac{b^2}{a^2}\,.\,2ax\,;$$

also, if $SC=c$,

$$b^2=a^2-c^2=(a+c)\,(a-c)\,;$$

and when C is taken infinitely distant from S, AC and SC, differing from each other only by the finite quantity AS, may be supposed equal, and we shall have therefore

$$a+c=2a,\ a-c=\mathrm{AS},$$

which values being substituted in the expression for y^2, we have

$$y^2=\frac{2a}{a^2}\,.\,2ax\,.\,\mathrm{AS}$$
$$=4\mathrm{AS}\,.\,x,$$

and this we know to be the equation to a parabola whose focus is S and vertex A.

33. Let the equation proposed be

$$\frac{y^2}{4h\,.\,\cos^2\alpha}-y\tan\alpha+x=0.$$

This may be put under the form

$$y^2-4h.\sin\alpha.\cos\alpha.y+4h.\cos^2\alpha.x=0,$$

and comparing it with equation (1) we find

$$\mathrm{A}=1,\ \mathrm{B}=0,\ \mathrm{C}=0,$$

so that

$$\mathrm{B}^2-4\mathrm{AC}=0,$$

and the curve is therefore a parabola, and we must get quit of y and the constants. For this purpose assume

$$x=x'+a,\ y=y'+b,$$

when we shall have

$$y'^2+2by'+b^2-4h\sin\alpha.\cos\alpha(y'+b)$$
$$+4h.\cos^2\alpha(x'+\alpha)=0,$$

in which the coefficient of y' is $2b-4h.\sin\alpha.\cos\alpha$, and the constant is $b^2-4h.\sin\alpha.\cos\alpha.b+4h.a\cos^2\alpha$:
Making these separately $=0$, we have first

$$b-2h.\sin\alpha.\cos\alpha=0,$$

whence

$$b=2h.\sin\alpha.\cos\alpha=h\sin 2\alpha\,;$$

substituting this value of b in the constant, and equating it to zero, we have

$$4h^2.\sin^2\alpha.\cos^2\alpha-8h^2\sin^2\alpha.\cos^2\alpha+4ha.\cos^2\alpha=0,$$

whence

$$a=h.\sin^2\alpha\,;$$

and therefore transferring the origin of the curve to a point, whose coordinates are $h.\sin^2\alpha$ and $h\sin 2\alpha$, the term involving y' and the constant part will vanish, and we shall have the equation

$$y'^2+4h\cos^2\alpha.\,x=0,$$

or

$$y'^2=-4h\cos^2\alpha.\,x,$$

which is that of a parabola, in which the Latus Rectum is $4h\cos^2\alpha$: the negative sign intimates that the values of x' are to be measured in a direction opposite to that in which the values of x were measured in the original equation; i. e. *downwards* instead of *upwards*. This result has been obtained without being under the necessity of changing the inclination of the axes; and it follows, therefore, that the new axes remain parallel to the former.

Cor. The equation discussed in this article is the one deduced to the path of a projectile (*Whewell's Mechanics*, art. 240), with this difference only, that x is put for y, and y for x; the reason of which is, that in the Mechanics x is measured along the horizontal line AR (fig. 19), whereas we measure x along the vertical axis A′B. If, therefore, A (fig. 19) be the point of projection and AA′R the path, it appears, from what has preceded, that this path is a parabola, the Latus Rectum of which is $4h\cos^2\alpha$, and vertex the point A′, determined by taking $AB=h\sin^2\alpha$, and $BA'=h\sin 2\alpha$. Also AB will be half the horizontal range, and BA′ the greatest altitude of the projectile above the horizontal line AR. Fig. 19.

Polar equations to the conic sections.

34. To find the polar equation to the parabola ASP (fig. 13), let Fig. 13.

$$AS=a,\ SP=u,\ \angle ASP=\theta:$$

then, by the property of the parabola, we have

$$\begin{aligned} SP &= PM \\ &= KN \\ &= KS+SN \\ &= 2AS+SP.\cos PSN \\ &= 2AS-SP.\cos ASP, \end{aligned}$$

whence

$$SP(1+\cos ASP)=2AS,$$

and

$$SP=\frac{2AS}{1+\cos ASP},$$

or

$$u=\frac{2a}{1+\cos\theta};$$

which is the polar equation required.

35. To obtain the polar equation to the ellipse APM (fig. 14), we have from the triangle SPH (Euclid, Book ii, Prop. 13) Fig. 14.

$$PH^2 = SP^2 + SH^2 - 2SH.SN;$$

and putting for PH its value (2AC − SP), and 2SC for SH,

$$(2AC - SP)^2 = SP^2 + (2SC)^2 - 4SC.SN,$$

or

$$4AC^2 - 4SP.AC + SP^2$$
$$= SP^2 + 4SC^2 - 4SC.SP.\cos PSN,$$

whence

$$AC^2 - SP.AC = SC^2 + SC.SP \cos ASP,$$

and

$$AC^2 - SC^2 = SP(AC + SC.\cos ASP),$$

and dividing each term by AC^2, this gives us

$$1 - \frac{SC^2}{AC^2} = \frac{SP}{AC}\left(1 + \frac{SC}{AC}\cos ASP\right).$$

Let now $AC = a$, $SP = u$, angle ASP $= \theta$, and the eccentricity $\frac{SC}{AC} = e$; then we have

$$1 - e^2 = \frac{u}{a}(1 + e.\cos\theta);$$

whence

$$u = \frac{a(1 - e^2)}{1 + e.\cos\theta},$$

the polar equation to the ellipse.

36. The polar equation to the hyperbola, deduced in a similar manner, is

$$u=\frac{a(e^2-1)}{1+e\cos\theta}.$$

37. The equation

$$u=\frac{a(1-e^2)}{1+e.\cos\theta}$$

may be considered as the general polar equation to the conic sections. It is the equation to the ellipse or the hyperbola, as e is less or greater than unity; it is the equation to the parabola when e is equal to unity, and the equation to the circle, when e is 0.

In the case of the parabola, when $e=1$, $1-e^2$ becomes 0, but at the same time a becomes infinite, and therefore

$$a(1-e^2)=\infty.0=\frac{1}{0}.0=\frac{0}{0},$$

the value of which may be any thing whatever; in the present case it may be shown, as in art. 32, that the true value is 2AS.

Oblique coordinates.

38. The axes of our coordinates have hitherto been supposed generally to be at right-angles to each other;
Fig. 20. but if we take for our axes the lines Ax', Ay' (fig. 20,) inclined at the oblique angle $x'Ay'$, it is evident that the position of the point P may be equally determined by means of the coordinates AM, MP; MP being supposed to be always drawn parallel to the axis Ay'.

39. Let Ax, Ay (fig. 20) be rectangular axes; Ax', Ay', oblique axes, inclined to Ax at angles θ and φ respectively; if then the equation to a curve be given between rectangular coordinates x, y, measured along Ax, Ay, it may be transferred to oblique coordinates x', y', measured along the axes Ax', Ay'. For let P be any point; AN, NP the rectangular coordinates, x, y; AM, MP the oblique coordinates x', y'; then, drawing MQ parallel to Ax, MO parallel to Ay or PN, we shall have

$$\begin{aligned}\mathrm{AN}=\mathrm{AO}+\mathrm{ON}=\mathrm{AO}+\mathrm{MQ}=\mathrm{AM}\cos x'\mathrm{A}x\\ +\mathrm{PM}\,.\cos\mathrm{PMQ},\\ \mathrm{PN}=\mathrm{PQ}+\mathrm{QN}=\mathrm{PQ}+\mathrm{MO}=\mathrm{PM}\sin\mathrm{QMQ}\\ +\mathrm{AM}\sin x'\mathrm{A}x,\end{aligned}$$

or

$$\begin{aligned}x&=x'\,.\cos\theta+y'\,.\cos\varphi,\\ y&=y'\,.\sin\varphi+x'\,.\sin\theta\,;\end{aligned}$$

and substituting these values of x, y, we shall have an equation between x', y', which will be the one required.

Hyperbola between the asymptotes.

40. Let PAP$'$ (fig. 21) be an hyperbola, whose semi- Fig. 21.
major and minor axes are a and b; its asymptotes are indefinite straight lines Cx', Cy', passing through the centre C and inclined to the axis Cx at equal angles x'Cx, y'Cx, the tangents of which are each equal to $\frac{b}{a}$.

If, now, the equation to the hyperbola be required, on the supposition of these asymptotes being the axes, making $\angle\, x'\mathrm{C}x=\theta$, we shall have $y'\mathrm{C}x=-\theta$, and substituting these for θ and φ in the formulæ of art. 39, the values of

x and y will be

$$x = x'.\cos\theta + y'.\cos-\theta = x'.\cos\theta + y'.\cos\theta$$
$$= (x' + y')\cos\theta,$$
$$y = x'.\sin\theta + y'.\sin-\theta = x'.\sin\theta - y'.\sin\theta$$
$$= (x' - y')\sin\theta;$$

whence the rectangular equation to the hyperbola

$$y^2 = \frac{b^2}{a^2}(x^2 - a^2)$$

will become

$$(x'^2 - 2x'y' + y'^2)\sin^2\theta = \frac{b^2}{a^2}\cos^2\theta\,(x'^2 + 2x'y' + y'^2) - b^2.$$

But, by hypothesis, $\tan\theta = \frac{b}{a}$, and therefore

$$\cos^2\theta = \frac{a^2}{a^2 + b^2},\ \sin^2\theta = \frac{b^2}{a^2 + b^2};$$

which values being substituted, we obtain

$$\frac{b^2}{a^2 + b^2}(x'^2 - 2x'y' + y'^2) = \frac{b^2}{a^2}\,\frac{a^2}{a^2 + b^2}(x'^2 + 2x'y' + y'^2) - b^2$$
$$= \frac{b^2}{a^2 + b^2}(x'^2 + 2x'y' + y'^2) - b^2;$$

and this being reduced, gives

$$4x'y' = a^2 + b^2,$$

whence

$$x'y' = \frac{a^2 + b^2}{4} = \textit{constant},$$

which is the equation to the hyperbola between the asymptotes. When the hyperbola is rectangular, we have $b = a$, and in this case therefore

$$x'y' = \frac{a^2}{2}.$$

Also since $\frac{b}{a} = 1$, we have $\tan \theta = 1$, so that the lines Cx', Cy', are each inclined to Ax at an angle of 45°, and are consequently at right angles to each other.

Method of determining the position of a point in space.

41. The points and lines of which we have hitherto spoken have been supposed to lie all of them in one plane, that of the coordinate axes Ax, Ay; and if therefore a point be given which does not lie in that plane, its position can no longer be determined by the previous methods.

In this case the point P (fig. 22) must first be referred to the plane of xy by a perpendicular PM let fall from P upon that plane, and the position of M be then fixed by two coordinates AN, NM, as before; and this method will evidently serve for any point whatever in space; for wherever the point be taken, its altitude above the plane of xy may always be expressed by a line such as PM, and the position of the point M, whence the perpendicular PM is to be drawn, be determined by two coordinates AN, NM. Fig. 22.

If, now, at the point A we erect the indefinite straight line Az at right angles to the plane of xy, the values of PM may be expressed by lines such as AQ measured along Az; and the values of NM being similarly represented by lines such as AO taken along Ay, AN, AO, AQ will be the three coordinates of the point P, and are denoted by x, y, z, respectively.

42. The three axes Ax, Ay, Az, being each of them produced indefinitely, if through the origin A we draw three planes passing, the first through x, y; the second through x, z; and the third through y, z; these planes will divide space into eight compartments, four lying above, and four below the plane of xy; and every point in space must lie in one or other of these compartments.

43. What was said respecting the signs of the two coordinates x, y, will apply also to the signs of the three x, y, z; thus if z be positive, it shows that the point in question is in one of the compartments lying above the plane of xy; but if z be negative, the point must be below that plane.

On the projections of a straight line in space, and its equations.

Fig. 23. 44. Let PP$''$ (fig. 23) be any straight line in space, and from each of its points P, P$'$, P$''$, &c. let fall on the plane of xy the perpendiculars PM, P$'$M$'$, P$''$M$''$, &c.; these will evidently all lie in one plane PM$''$, which is called the *projecting* plane; and the section of this plane with that of xy will be the straight line MM$''$, formed by the feet of the perpendiculars. This line is the *projection* of PP$''$ on the plane of xy, and by drawing perpendiculars in a similar manner to the planes of xz, yz, we shall obtain the *projections* of PP$''$ on those planes.

45. The projecting planes must, from the construction, contain both the line proposed and its projection; and two of these projections will consequently serve to determine the line; for if two of the projections be given,

we shall be able to draw two projecting planes, each of which will contain the line required, and that line, therefore, will be the intersection of the two planes.

46. If now in the line PP″ we take any point P whose coordinates are x, y, z, and let fall the perpendicular PM, the coordinates x, y, will obviously be the same both for the point P and for its projection M; and since the same may be said of all corresponding points in the line and its projection, the coordinates x, y, of the line PP″ will be connected together by the same equation that belongs to the projection MM″; and these observations will apply equally to the other projections. But these projections being straight lines lying wholly in the planes of xy, xz, and yz, respectively, their equations will be of the forms

$$y = ax + \alpha,$$
$$z = bx + \beta,$$
$$y = cz + \gamma,$$

and these therefore will be the equations which connect the coordinates x, y, z, of the line PP″, and are consequently said to be the equations of that line.

Any two of the above equations will be sufficient; for two being given we can from them derive two of the projections; and, as was shown (art. 44), two of the projections will serve to determine the line.

Equation to a plane.

47. Let DBC (fig. 24) be a plane cutting the axis of z in the point B, and the coordinate planes xz, yz, in the lines BC, BD; take in it any point P, whose coordinates Fig. 24.

are x, y, z; draw, in the plane DBC, PQ parallel to BC and cutting DB in Q, and in the plane yz draw QO parallel to Az, BF parallel to Ay; also let QE be perpendicular to PM, and therefore parallel to Ax: then we have

$$\begin{aligned}\mathrm{PM}=\mathrm{PE}+\mathrm{EM}&=\mathrm{PE}+\mathrm{QO}\\ &=\mathrm{PE}+\mathrm{QF}+\mathrm{FO}\\ &=\mathrm{QE}.\tan\mathrm{PQE}+\mathrm{BF}.\tan\mathrm{QBF}+\mathrm{AB}\\ &=\mathrm{AN}.\tan\mathrm{PQE}+\mathrm{AO}.\tan\mathrm{QBF}+\mathrm{AB},\end{aligned}$$

whence

$$z=x.\tan\mathrm{PQE}+y.\tan\mathrm{QBF}+\mathrm{AB};$$

but PQE and QBF being the angles which the sections BC, BD make with the axes of x, y, respectively, they will remain the same for any point P in the plane, and their tangents may therefore be represented by the constant coefficients a, b; AB is also constant, and may be represented by c; when our equation will become

$$z=ax+by+c,$$

which is the equation to the plane DBC.

This is a general simple equation between three variables, and by assuming the coefficients properly, may be put under the form

$$\mathrm{A}x+\mathrm{B}y+\mathrm{C}z+\mathrm{D}=0.$$

When y is made $=0$, this becomes

$$\mathrm{A}x+\mathrm{C}z+\mathrm{D}=0,$$

which is the equation to the section BC; and similarly by making $x=0$, we shall have

$$By+Cz+D=0,$$

for the equation to the section BD.

If the plane pass through the origin A, the constant D will disappear, and the equation be of the form

$$Ax+By+Cz=0.$$

Equation to the sphere.

48. Let α, β, γ, be the coordinates A*a*, *ab*, *b*C, of the centre C of the sphere (fig. 25); x, y, z, the coordinates of any other point P on the surface of the sphere; r the radius CP: draw C*n* perpendicular PM, and *bm* perpendicular MN; then we have Fig. 25.

$$\begin{aligned}CP^2=Cn^2+Pn^2&=bM^2+Pn^2\\&=bm^2+Mm^2+Pn^2;\end{aligned}$$

or

$$r^2=(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2;$$

which is the equation to the sphere.

Equation to the cone.

49. Let ABD (fig. 26) be a cone, in which the axis AC$=a$, the radius CD of the base$=b$; take C the centre of the base for the origin of the coordinates, and let P be any point in the surface whose coordinates are CN, NM, MP: draw PQ perpendicular to AC, and consequently equal to CM: then Fig. 26.

$$PQ=CM=\sqrt{CN^2+NM^2}=\sqrt{x^2+y^2};$$

but from the similar triangles APQ, ADC, we have

$$PQ : AQ :: CD : AC,$$

or

$$\sqrt{x^2+y^2} : a-z :: b : a,$$

whence

$$a\sqrt{x^2+y^2}=ab-bz,$$

and therefore

$$bz=ab-a\sqrt{x^2+y^2},$$

$$z=\frac{a}{b}(b-\sqrt{x^2+y^2}),$$

the equation to the cone required.

ERRATA.

Page *xiii.* line 8 from the bottom, for $AM=a$, read $AB=a$.
xxi. last line, for *ordinates,* read *co-ordinates.*
xxxiii. line 2 from the bottom, for *director,* read *directrix.*
xxxvii. line 6 from the bottom, for *mean proportional between,* read *third proportional to.*
xl. line 3 from the bottom, for *ellipse,* read *cone.*
xlv. line 6 from the bottom, for $+2x'y'$, read $-2x'y'$; and for y'^2, read x'^2: and y'^2 for x'^2.
line 4 from the bottom, for $+C\,Sin.\,2\theta$, read $-C\,Sin.\,2\theta$.
last line, for $+C\,tan.\,2\theta$, read $-C\,tan.\,2\theta$.
xlvi. line 2, for $A+C$, read $A-C$.
liii. line 12, for QMQ, read PMQ.

Page 8, line 5 from the bottom, after $tdu+udt$, insert (8).
11, line 17, after $x+h$, insert *in* $y=x^m$.
line 3 from the bottom, after $\frac{y'-y}{h}$ insert $=$.
14, line 8, for *elimination,* read *differentiation.*
26, line 8, after $\frac{Sin.\,h}{Cos.\,h+1}$, insert $\frac{Sin.\,h}{h}$.
39, line 15, for $\frac{dy}{dx}dy$, read $\frac{dy}{dx}dx$.
43, line 14, for y^2, read y'^2.
44, line 12 from the bottom, for $2y$, read $2y'$.
46, line 14 from the bottom, insert (35).
53, line 18, transpose B^2 and B.
63, line 5 from the bottom, for $2\pi\frac{b^2}{a^2}$, read $2\pi\frac{b^2}{a}$.
68, line 1, for *minimum,* read *maximum.*
75, line 12, for SN, read $M''N$.
line 13, for $M''N$, read SN.
line 16, for $\frac{dy}{dx^2}$, read $\frac{d^2y}{dx^2}$.
84, line 9, for $a+h$, read $a-h$.
89, line 3 from the bottom, for $\frac{dy^2}{dx^2}$, read $\frac{d^2y}{dx^2}$.
94, line 8 from the bottom, for $=0$, read $=\frac{0}{0}$.
95, line 8 from the bottom, for $\varphi x'=$, read $\varphi x'+$.
102, line 5, insert (73).
104, line 4 from bottom, for dx^2+dy, read dx^2+dy^2.

Page 110, line 2 from the bottom, for $y+s=0$, read $y+s=\textit{constant}$.
121, lines 10, 13, 15, and 17, for $\frac{x}{y}$, read $\frac{y}{x}$.
122, line 7, for $\frac{x}{y}$, read $\frac{y}{x}$, in two places.
123, line 8 from the bottom, for *At*, read *Pt*.
125, line 16, for $\frac{d\theta}{du}$, read $\frac{du}{d\theta}$.
128, line 4, for 107, read 108.
line 8, for m *Cos.* θ, read m *Sin.* θ.
line 10, for m *Cos.* θ, read m *Sin.* θ.
129, line 5, for $\frac{}{2\pi}$, read $\frac{\theta}{2\pi}$.
131, line 8, in the denominator, for $u^2d\theta^2$, read $u^2d\theta^3$.
139, line 2, for *cord*, read *chord*.
141, line 10 from the bottom, for $M'E$, read $M'F'$.
144, line 4, in the denominator, for $4a^2x^2$, read $1+4a^2$.
154, line 14, for M, read N.
163, line 6, for Mh, read Mh^n.
164, line 12, for $\frac{d^{n+1}f(a+h)}{da^{n+1}}+$, read $\frac{d^{n+1}f(a+h)}{da^{n+1}}=$.
185, line 1, for *take the series*, read *integrate by series*.
194, line 1, in the denominator, for P, read P'; and for S, read S'.
195, line 11, *dele* $\int$.
204, last line, for Ba^2, read $2Ba^2$.
213, line 5 from the bottom, *dele* $\int$.
216, line 5 from the bottom, for $(x-d)^n$, read $(x-f)^n$.
220, line 11, for B, read b.
221, line 10, for $\sqrt{m^2+z^2}$, read $\sqrt{m^2+x^2}$; and $4z^3$, for $4z$.
224, line 3, for $\frac{8a^2z^2dz}{(z^2+1)^2}$, read $\frac{8a^2z^2dz}{(z^2+1)^3}$.
227, line 5, for $\frac{a}{x}$, read $\frac{a}{x}n$.
228, line 8, insert (52).
229, line 11 from the bottom, for $(a+bx^{np})$, read $(a+bx^n)^p$.
234, line 7, after $=$, insert $-$.
239, line 2 from the bottom, for $\sqrt{1-z^2}$, read $1-z^2$.
241, line 6 from the bottom, insert).
245, line 4, for *replacing the process*, read *repeating the process*.
249, line 7, add (73).
252, line 5, for $\frac{b}{a}$, *area*, read $\frac{b}{a}\times$ *area*.
256, line 5 from the bottom, for dx^2dy^2, read dx^2+dy^2.

Page 263, line 3, dele *(Note seventh)*

266, line 4 from the bottom, for *plane tangent*, read *tangent-plane;* and for y, read xy.

line 2 from the bottom, for $\sqrt{1+\left(\frac{dz}{dy}\right)^2+\left(\frac{dz}{dx}\right)^2}$, read $\frac{1}{\sqrt{1+\left(\frac{dz}{dy}\right)^2+\left(\frac{dz}{dx}\right)^2}}$.

268, line 2 from the bottom, for *axis*, read *axes*.

272, last line, dele *since*.

276, line 9, for xp, read x^p.

line 5 from the bottom, for $Mx^r y_s + Nx_{r'} y^{s'}$, read $Mx^r y^s + Nx^{r'} y^{s'}$.

278, line 7 from the bottom, for $(t, u)x$, read $(t, u)dx$.

line 5 from the bottom, after xdu, put).

290, line 11 from the bottom, for $Mdx + Ndy$, read $zMdx + zNdy$.

299, line 15 from the bottom, dele $=0$.

line 13 from the bottom, insert $=0$.

306, line 8 from the bottom, for $\frac{d^{n-1}y}{dx^n}$, read $\frac{d^{n-1}y}{dx^{n-1}}$.

324, line 4 from the bottom, for Nz, read Nx.

328, line 12, for $\frac{d^2y}{dx^2}$, read $\frac{d^2y}{dx}$; and for c read C.

353, line 7 from the bottom, for $z=mx$, read $z=mx^2$.

355, line 7, for *the function*, read *a function*.

line 13 from the bottom, for $\frac{dz^2}{dx^2}$, read $\frac{d^2z}{dx^2}$.

361, line 10, for $\frac{h^4}{4x}$, read $\frac{h^4}{x^4}$.

377, line 10, for zy, read xy.

378, line 6, for $\frac{C}{A}$, read $\frac{A}{C}$.

397, line 2, for (24), read (4).

In pages 29, 34, 134, 175, 181, 184, 185, 188, 189, 190, 192, 195, 196, 197, 200, 203, 204, 205, 207, 211, 212, 235, 241, 242, 243, 276, 277, 291,—for *log.* read *L*.

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ELEMENTS

OF THE

DIFFERENTIAL AND INTEGRAL CALCULUS.

DIFFERENTIAL CALCULUS.

On the differentiation of algebraic quantities.

1. One variable is said to be a function of another variable, when the first is equal to a certain analytical expression composed of the second; for example, y is a function of x in the following equations:

$$y=\sqrt{a^2-x^2},\ y=x^3-3bx^2,\ y=\frac{x^2}{a},\ y=b+cx^3.$$

2. Let us consider a function when in its state of increase, by reason of the increase of the variable which it contains; and since every function of a variable x may be represented by the ordinate of a curve BMM′, fig. 1, let $AP=x$ and Fig. 1.
$PM=y$ be the coordinates of a point M in that curve, and suppose that the abscissa AP receives an increment $PP'=h$; then the ordinate PM will become $P'M'=y'$. In order, therefore, to obtain the value of this new ordinate, we see that we must change x into $x+h$ in the equation of the curve, and the value which the equation shall then determine for y will be that of y'.

For example, if we had the equation $y=mx^2$, we should obtain y' by changing x into $x+h$, and y into y', and we should have

$$y'=m(x+h)^2,$$

or, by developing,

$$y'=mx^2+2mxh+mh^2.$$

3. Let us take also the equation

$$y=x^3,\ \ldots\ldots\ (1),$$

and suppose that when x becomes $x+h$, y becomes y', we shall have then

$$y'=(x+h)^3\ ;$$

or, by expanding,

$$y'=a^3+3x^2h+3xh^2+h^3:$$

if from this equation we subtract equation (1) there will remain

$$y'-y=3x^2h+3xh^2+h^3,$$

and by dividing by h,

$$\frac{y'-y}{h}=3x^2+3xh+h^2\ \ldots\ (2).$$

Let us see now what is to be learnt from this result: $y'-y$, being the difference between the new value of y and its primitive one, represents the increment of the function y in consequence of the increment h given to x; and the increment of x, on the other hand, being h, it follows that the expression $\frac{y'-y}{h}$ is the ratio of the increment of the function y to that of the variable x. By attending to the second side of equation (2), we see that this ratio is diminished the more h is diminished, and that when h becomes 0 this ratio is reduced to $3x^2$. This term $3x^2$ is therefore the limit of the ratio $\frac{y'-y}{h}$, being the term to which it tends as we diminish h.

4. Since, on the hypothesis of $h=0$, the increment of y becomes also 0, $\frac{y'-y}{h}$ is reduced to $\frac{0}{0}$, and consequently the equation (2) becomes

$$\frac{0}{0}=3x^2.$$

This equation involves in it nothing absurd, for from Algebra we know that $\frac{0}{0}$ may represent every sort of quantity; besides which it will be easily seen, that since by dividing the two terms of a fraction by the same number the fraction is not altered in value, it follows that the smallness of the terms of a fraction does not at all affect its value, and that, consequently, it may remain the same when its terms are diminished to the last degree, that is to say, when they become each of them 0.

The fraction $\frac{0}{0}$ which appears in the equation (3), is a symbol which has expressed the ratio of the increment of the function to that of the variable: since this symbol retains no trace of that variable, we will represent it by $\frac{dy}{dx}$; and then $\frac{dy}{dx}$ will remind us that the function was y and the variable x; but dy and dx will be no less evanescent quantities, and we shall have

$$\frac{dy}{dx}=3x^2 \quad \ldots\ldots (4).$$

$\frac{dy}{dx}$, or rather its value $3x^2$ is the differential coefficient of the function y.

We may observe that $\frac{dy}{dx}$ being the symbol which represents the limit $3x^2$, [as is shown by equation (4)], dx ought properly to be always placed under dy. In order, however, to facilitate operations in algebra, we may for a time clear equation (4) of its denominator, and we shall have $dy=3x^2dx$. The expression $3x^2dx$ is what we call the differential of the function y.

5. Let us seek also the differential of the function $a+3x^2$; for which purpose we must, in the equation $y=a+3x^2$, make $x=x+h$; and changing y into y', the equation will become

$$y'=a+3x^2+6xh+3h^2;$$

therefore

$$\frac{y'-y}{h}=6x+3h,$$

and making $h=0$, there results $\frac{dy}{dx}=6x$; the differential sought therefore is $dy=6xdx$.

6. For a third example, let us seek the differential of $y=ax^3-b^3$; making $x=x+h$, and substituting, we have

$$y'=ax^3+3ax^2h+3axh^2+ah^3-b^3;$$

therefore

$$\frac{y'-y}{h}=3ax^2+3axh+ah^2,$$

and taking the limit, we have

$$\frac{dy}{dx}=3ax^2.$$

This is the differential coefficient of the proposed function; the differential will be $dy=3ax^2dx$.

7. Let it be proposed to find also the differential of $y=\frac{1-x^3}{1-x}$: performing the division we find $y=1+x+x^2$; putting $x+h$ in place of x and y' in that of y, we obtain

$$y'=1+x+h+x^2+2xh+h^2;$$

and arranging according to the powers of h,

$$y'=1+x+x^2+(2x+1)h+h^2;$$

therefore,

$$\frac{y'-y}{h}=2x+1+h;$$

taking the limit, we have $\frac{dy}{dx}=2x+1$; and therefore the differential of $\frac{1-x^3}{1-x}$ is $(2x+1)\,dx$.

8. As another example let us take

$$y=(x^2-2a^2)\,(x^2-3a^2)\,;$$

developing, we have

$$y=x^4-5a^2x^2+6a^4\,;$$

putting $x+h$ for x and y' for y, and arranging them according to the powers of h, there results,

$$y'=x^4-5a^2x^2+6a^4+(4x^3-10a^2x)h$$
$$+(6x^2-5a^2)h^2+4xh^3+h^4\,;$$

therefore

$$\frac{y'-y}{h}=4x^3-10a^2x+\,(6x^2-5a^2)\,h+4xh^2+h^3\,;$$

passing to the limit, we have

$$\frac{dy}{dx}=4x^3-10a^2x\,;$$

and multiplying by dx, we find that the differential is

$$dy=(4x^3-10a^2x)\,dx.$$

9. The expression dx is itself the differential of x; for let $y=x$; we have then $y'=x+h$; therefore $y'-y=h$, and consequently $\frac{y'-y}{h}=1$; and since the quantity h does not enter into the second side of this equation, we see that to pass to the limit it is sufficient to change $\frac{y'-y}{h}$ into $\frac{dy}{dx}$; which gives $\frac{dy}{dx}=1$, and therefore $dy=dx$.

10. We should find, similarly, that the differential of ax is adx; but if we had $y=ax+b$, we should still obtain adx for the differential: whence it follows, that a constant b, which

is not affected by x, gives no term to the differentiation, or, in other words, has no differential.

We might, besides, consider, that if we have $y=b$, it is the case in which a is 0 in the equation $y=ax+b$, and in which, consequently, $\frac{dy}{dx}=a$ is reduced to $\frac{dy}{dx}=0$, so that there is neither limit nor differential.

11. It may be observed, that sometimes the increment of the variable is negative; in which case we must put $x-h$ for x, and proceed as before.

Thus, to find the differential of ax^3 when the increment is negative, we must replace x by $x-h$, and we shall have

$$y'=ax^3-3ax^2h+3axh^2-ah^3;$$

therefore,

$$\frac{y'-y}{h}=-3ax^2+3axh-ah^2:$$

taking the limit, we shall have $\frac{dy}{dx}=-3ax^2$, and, consequently, $dy=-3ax^2dx$.

We see that this comes to the same thing as supposing dx negative in the differential of y calculated on the hypothesis of a positive increment.

12. Before proceeding farther, we must make one essential remark; viz., that in an equation, of which the second side is a function of x, and which, for that reason, we will represent generally by $y=fx$, if on changing x into $x+h$, and arranging the terms according to the powers of h, we find the following development:

$$y'=\mathrm{A}+\mathrm{B}h+\mathrm{C}h^2+\mathrm{D}h^3+\&c.,$$

we ought always to have $y=\mathrm{A}$.

For if we make $h=0$, the second side is reduced to A: in regard to the first side, since we have accented y only, to indicate that y has undergone a certain change on x becoming $x+h$, it follows necessarily, that when h is 0, we must suppress the accent of y, and the equation will be reduced then to $y=\mathrm{A}$.

13. This will give us the means of generalizing the process of differentiation. For, if in the equation $y=fx$, in which we are supposed to know the expression represented by fx, we have put $x+h$ in place of x; and after having arranged the terms according to the powers of h, are able to obtain the following development:

$$y'=\mathrm{A}+\mathrm{B}h+\mathrm{C}h^2+\mathrm{D}h^3+\&c.,$$

or rather, according to the preceding article,

$$y'=y+\mathrm{B}h+\mathrm{C}h^2+\mathrm{D}h^3+\&c.,$$

we shall have,

$$y'-y=\mathrm{B}h+\mathrm{C}h^2+\mathrm{D}h^3+\&c.;$$

therefore,

$$\frac{y'-y}{h}=\mathrm{B}+\mathrm{C}h+\mathrm{D}h^2+\&c.:$$

and taking the limit,

$$\frac{dy}{dx}=\mathrm{B};$$

which shows us that the differential coefficient is equal to the coefficient of the term which contains the first power of h, in the development of $f(x+h)$, arranged according to the ascending powers of h.

14. If instead of one function y, which changes its value in consequence of the increment given to the variable x, which it contains, we have two functions, y and z, of that same variable x, and we know how to find separately the differentials of each of these functions, it will be easy, by the following demonstration, to determine the differential of the product zy of those functions. For if we substitute $x+h$ in place of x, in the functions y and z, we shall obtain two developments, which, being arranged according to the powers of h, may be represented thus,

$$y'=y+\mathrm{A}h+\mathrm{B}h^2+\mathrm{C}h^3+\&c. \ldots\ldots (5),$$

$$z'=z+\mathrm{A}'h+\mathrm{B}'h^2+\mathrm{C}'h^3+\&c. \ldots\ldots (6);$$

Passing to the limit, we shall find

$$\frac{dy}{dx}=\text{A},\ \frac{dz}{dx}=\text{A}' \ldots\ldots\ldots\ldots\ldots\ldots (7);$$

multiplying equations (5) and (6) the one by the other, we shall obtain

$$\begin{aligned} z'y' = zy &+ \text{A}zh + \text{B}zh^2 + \&\text{c.} \\ &+ \text{A}'yh + \text{AA}'h^2 + \&\text{c.} \\ &\qquad\qquad + \text{B}'yh^2 + \&\text{c.}; \end{aligned}$$

therefore,

$$\frac{z'y'-zy}{h}=\text{A}z+\text{A}'y+(\text{B}z+\text{AA}'+\text{B}'y)\,h+\&\text{c.};$$

and taking the limit, and indicating, by a point placed before it, the expression to be differentiated, we shall get

$$\frac{d.\,zy}{dx}=\text{A}z+\text{A}'y;$$

putting, in place of A and A′, their values, given by equations (7), there will result,

$$\frac{d.\,zy}{dx}=z\frac{dy}{dx}+y\frac{dz}{dx},$$

and suppressing the common factor dx,

$$d.\,zy=zdy+ydz.$$

Thus, to find the differential of the product of two variables, we must multiply each by the differential of the other, and add the products.

15. By means of this rule, we shall easily find the differential of a product of three variables.

Let it, for example, be yzu; make $yz=t$, when we shall have $d.\,yzu=d.\,tu$.

But by what has preceded,

$$d.\,tu=tdu+udt,$$

and since $t=yz$, we have $dt=ydz+zdy$.

Substituting, therefore, these values of t and dt, in equation (8), it is changed into

$$d.\,yzu=yzdu+uydz+uzdy.$$

We see, then, that the same rule still holds for a product of three variables; viz., *that we must write down the product yzu, replace successively each variable by its differential, and add the products.*

The same rule holds good for any greater number of variables.

16. The differential of a fraction $\frac{z}{y}$ is $\frac{ydz-zdy}{y^2}$; for suppose $\frac{z}{y}=t$, we have $z=ty$, and $dz=tdy+ydt$ (art. 14), from which we find $ydt=dz-tdy$:
putting on the second side the value of t, there results,

$$ydt=dz-\frac{z}{y}dy,$$

reducing to the same denominator,

$$ydt=\frac{ydz-zdy}{y},$$

and lastly,

$$dt=\frac{ydz-zdy}{y^2}, \text{ or } d.\frac{z}{y}=\frac{ydz-zdy}{y^2}.$$

17. If in the equation $d.yzu=yzdu+yudz+uzdy$ (art. 15), we divide each term by yzu, we shall get

$$\frac{d.yzu}{yzu}=\frac{du}{u}+\frac{dz}{z}+\frac{dy}{y};$$

and generally, by dividing the differential of the product of any number of variables by the product itself, we shall find,

$$\frac{d.xyztu\ldots\ldots}{xyztu\ \ldots\ldots}=\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}+\frac{dt}{t}+\frac{du}{u}+\ldots (9).$$

If now x, y, z, t, u, &c., be each equal to x, and the number of them be m, we shall have on the second side of this equation m terms, each equal to $\frac{dx}{x}$; the second side will therefore be changed into $\frac{mdx}{x}$, and the equation (9) will become

$$\frac{d.x^m}{x^m} = \frac{mdx}{x},$$

and multiplying by x^m, we shall have

$$d.x^m = mx^{m-1}dx,$$

18. We may hence deduce this rule: *when a variable is raised to a power m, to obtain its differential, we must* 1°. *make the index the coefficient;* 2°. *diminish the index of the variable by unity;* 3°. *multiply this product by dx.*

19. The same rule will hold, if the index be fractional or negative.

To prove the first case, let $y = x^{\frac{p}{q}}$; raise both sides to the power q, when we shall have $y^q = x^p$, and therefore, art. 18, $qy^{q-1}dy = px^{p-1}dx$; whence we find,

$$dy = \frac{p}{q}.\frac{x^{p-1}}{y^{q-1}}dx;$$

and since x^{p-1}, y^{q-1}, may be put under the forms $\frac{x^p}{x}$, $\frac{y^q}{q}$, substituting these values, we have

$$dy = \frac{p}{q}.\frac{x^p}{y^q}.\frac{y}{x}dx;$$

and since $x^p = y^q$, the preceding equation is reduced to

$$dy = \frac{p}{q}\frac{y}{x}dx;$$

Lastly, putting for y its value, we obtain

$$dy = \frac{p}{q}.\frac{x^{\frac{p}{q}}}{x}dx;$$

and bringing the denominator x into the numerator, we have

$$dy = \frac{p}{q}.x^{\frac{p}{q}-1}\quad dx,$$

a result the same as would have been obtained for the differential of $y = x^{\frac{p}{q}}$ by applying the rule given Art. 18.

To demonstrate the case in which the index is negative, let $y = x^{-p}$: this is the same with $y = \frac{1}{x^p}$, which being differentiated by the rule for fractions, art. 16, we shall have

$$dy = \frac{x^p d.1 - 1.dx^p}{x^p \times x^p}.$$

Observing that unity being a constant, its differential is 0, art. 10, this expression is reduced to $dy = -\frac{d.x^p}{x^{2p}}$; whence by differentiating, art. 18, we shall have $dy = -\frac{px^{p-1}dx}{x^{2p}}$, and subtracting the index $2p$ from the index $p-1$, there results lastly $dy = -px^{-p-1}dx$, as we should have found by applying the rule of art. 18. We conclude, therefore, that this rule holds true, whatever be the index of x, that is to say, whether the index be integral, fractional, positive or negative.

20. We may arrive immediately at the differential of x^m by means of the binomial, in the manner following;

making $x = x + h$, we obtain

$$y' = (x + h)^m,$$

and developing by the binomial theorem, we find

$$y' = x^m + mx^{m-1}h + m.\frac{m-1}{2}\,x^{m-2}h^2$$

$$+ m.\frac{m-1}{2}.\frac{m-2}{3}\,x^{m-3}h^3 +, \&c.$$

subtracting from this the preceding equation, and dividing by h, there remains

$$\frac{y'-y}{h}mx^{m-1} + m.\frac{m-1}{2}\,x^{m-2}\,h + m.\frac{m-1}{2}.\frac{m-2}{3}x^{m-3}h^2 + \&c.$$

Passing to the limit, by making $h = 0$, we obtain

$$\frac{dy}{dx} = mx^{m-1}\text{; therefore, } dy = mx^{m-1}dx\text{;}$$

and replacing the value of y, we have

$$d.x^m = mx^{m-1}dx.$$

21. If we replace the radical signs by fractional indices, the rule of Art. 18 will then serve to differentiate quantities of that sort. For example, to find the differential of $\sqrt{z}$, we must write $z^{\frac{1}{2}}$, the differential of which will be $\frac{1}{2}z^{-\frac{1}{2}}dz = \frac{dz}{2\sqrt{z}}$; which shows us, that *to obtain the differential of the square root of a variable quantity, we must divide the differential of that quantity by the double of its square root.*

On the differentiation of a sum of functions.

22. The process for differentiating a quantity which contains several terms would be exceedingly long, were it necessary always to pursue the course we have hitherto followed; finding first the value of y', in order to deduce from it that of $\frac{y'-y}{h}$, and then passing to the limit by making $h=0$. Fortunately, if we can differentiate each term separately, we may adopt a more simple course by means of a theorem, which may thus be expressed: *the differential of a sum of functions is equal to the sum of the differentials of those functions.*

To demonstrate this, let f, F, φ, &c., be the symbols of the different functions of which y is composed, and suppose we have

$$y = fx + \mathrm{F}x + \varphi x + \&c.,$$

of which it is required to find the differential.

If we put $x+h$ for x in each of these functions, since by hypothesis we know how to develop each of them separately, according to the powers of h, we shall be able to express the result by

$$\begin{aligned} y' = fx &+ \mathrm{A}h + \mathrm{A}'h^2 + \&c. \\ + \mathrm{F}x &+ \mathrm{B}h + \mathrm{B}'h^2 + \&c. \\ + \varphi x &+ \mathrm{C}h + \mathrm{C}'h^2 + \&c. \end{aligned}$$

And collecting the terms multiplied by the same powers of h, and subtracting y, we shall find

$$y'-y=(A+B+C)h+(A'+B'+C')h^2+\&c. ;$$

and taking the limit,

$$\frac{dy}{dx}.=A+B+C, \quad dy=Adx+Bdx+Cdx.$$

But A, B, C, are the terms multiplied by the first power of h, in the development of $f(x+h)$, $F(x+h)$ $\varphi(x+h)$, whence it follows, that $Adx+Bdx+Cdx$ represents the sum of the differentials of the proposed function.

23. To give an application of this theorem, suppose that we have to find the differential of

$$y=ax^3+b^2x^2+c^4\sqrt{x} ;$$

we know, by article 10, that the differential of ax^3 is $ad.x^3$, and by differentiating according to article 18, and putting the numerical coefficient first, we obtain $3ax^2dx$. Following the same plan in respect of the constant b^2, we shall find that the differential of b^2x^2 is $2b^2xdx$; and the article 21 shows us, that $c^4\sqrt{x}$ has for its differential $\frac{c^4dx}{2\sqrt{x}}$. Adding, then, these results, we shall find

$$dy=3ax^2dx+2b^2xdx+\frac{c^4dx}{2\sqrt{x}}.$$

24. In general, when in an expression which we wish to differentiate, a constant appears as a factor of a function of x, we must differentiate as though there were no constant, and then multiply by the constant.

25. If, on the contrary, the constant be not connected with a function of x, it, as we have seen, article 10, will give no additional term to the differential.

On the manner of facilitating the differentiation of complicated functions, and of avoiding the process of elimination,

when the function y is not given immediately in terms of the variable x.

26. Sometimes the function y and the variable x are not given by a single equation. For example, if we had two equations of the forms $y=fu$, and $u=\varphi x$, the first mode that would present itself for obtaining the differential coefficient $\frac{dy}{dx}$ would be to eliminate u betwixt the two equations, so that we might apply the process of elimination; but without having recourse to this preliminary operation, we may obtain immediately the differential coefficient $\frac{dy}{dx}$, which will be the object in view in the following demonstration:

Suppose that when in the equation $u=\varphi x$, we put $x+h$ for x, u becomes $u'=u+k$; and that when we put $u+k$ for u in the equation $y=fu$, the function y becomes y'; if then by developing the functions of u and x according to the powers of their respective increments, the substitution of $x+h$ for x in the function u gives us

$$u'=u+qh+q'h^2+q''h^3+\text{\&c.};$$

and the substitution of $u+k$ for u in the function y gives us

$$y'=y+pk+p'k^2+p''k^3+,\text{\&c.}$$

we shall obtain from these equations

$$\left.\begin{aligned}\frac{u'-u}{h}&=q+q'h+q''h^2+\text{\&c.}\\ \frac{y'-y}{k}&=p+p'k+p''k^2+\text{\&c.}\end{aligned}\right\}\quad\ldots\ (10).$$

and multiplying the equations together, we shall have

$$\frac{u'-u}{h}\cdot\frac{y'-y}{k}=(p+p'k+p''k^2+\text{\&c.})\,(q+q'h+q''h^2+\text{\&c.})$$

The first side of this equation may be reduced; for the increment of u being represented by k, is therefore equal to

$u'-u$, and, consequently, instead of $\frac{y'-y}{k}\cdot\frac{u'-u}{h}$ we may write $\frac{y'-y}{h}$; when, putting $x'-x$ in place of h, the preceding equation becomes

$$\frac{y'-y}{x'-x}=(q+q'h+q''h^2+\text{\&c.})\,(p+p'k+p''k^2+\text{\&c.})\;\ldots\;(11).$$

When h is 0, k also vanishes (since u received its increment only because x became $x+h$), and, therefore, in the case of $h=0$, which is that of the limit, the equation (11) becomes

$$\frac{dy}{dx}=pq\;\ldots\ldots\;(12).$$

For the determining of p and q, we must make h and k each $=0$, in the equations (10), when those equations give us

$$\frac{dy}{du}=p,\ \frac{du}{dx}=q;$$

and substituting these values of p and q in equation (12), we obtain

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}\;\ldots\ldots\;(13).$$

This result shows us that if we have two equations, $y=fu$ and $u=\phi x$, and we find the values of the differential coefficients $\frac{dy}{du}$ and $\frac{du}{dx}$, the multiplying these two together will give us the value of $\frac{dy}{dx}$.

27. If, for example, we have the equations $y=3u^2$ and $u=x^3+ax^2$, we shall find

$$\frac{dy}{du}=6u,\ \frac{du}{dx}=3x^2+2ax;$$

and therefore, multiplying these equations together, we shall have

$$\frac{dy}{dx}=6u\,(3x^2+2ax)=6\,(x^3+ax^2)\,(3x^2+2ax).$$

28. The formula (13) is one of great use in differentiating complicated functions; we will give a few applications of it.

1°. Let it be required to find the differential of $y=\sqrt{a^2-x^2}$.

This will be done by finding the differential co-efficient $\frac{dy}{dx}$; for which purpose let $a^2-x^2=u$, and therefore . . . . . $y=\sqrt{u}=u^{\frac{1}{2}}$; then the equations $y=fu$ and $u=\varphi x$ (art. 26) will be here represented by

$$y=u^{\frac{1}{2}},\ u=a^2-x^2.$$

Differentiating these equations (art. 18) we find

$$\frac{dy}{du}=\tfrac{1}{2}.u^{-\frac{1}{2}}=\tfrac{1}{2}\ (a^2-x^2)^{-\frac{1}{2}},\ \frac{du}{dx}=-2x\,;$$

multiplying these differential co-efficients together, we obtain

$$\frac{dy}{dx}=-x\ (a^2-x^2)^{-\frac{1}{2}}=\frac{-x}{\sqrt{a^2-x^2}},$$

and therefore

$$dy=-\frac{xdx}{\sqrt{a^2-x^2}}.$$

Again let $y=\ (a+bx^m)^n$; to find the differential, making $a+bx^m=u$, we shall have the equations $y=u^n$, $u=a+bx^m$; therefore

$$\frac{dy}{du}=nu^{n-1}=n\ (a+bx^m)^{n-1},\ \frac{du}{dx}=bmx^{m-1}\,;$$

and multiplying these differential co-efficients together, we have

$$\frac{dy}{dx}=bmnx^{m-1}(a+bx^m)^{n-1}.$$

29. As a third example, let

$$y=\left(a+\sqrt{b-\frac{c}{x^2}}\right)^4,$$

suppose

$$b-\frac{c}{x^2}=u\ .\ .\ .\ .(14)\,;$$

and therefore

$$y = (a+\sqrt{u})^4 \quad \ldots . (15).$$

Differentiating equation (14), we have

$$du = \frac{2cxdx}{x^4}$$

and therefore

$$\frac{du}{dx} = \frac{2c}{x^3};$$

the equation (15) gives

$$dy = 4\,(a+\sqrt{u})^3 d.\,(a+\sqrt{u}) = 4\,(a+\sqrt{u})^3.\,\frac{du}{2\sqrt{u}}:$$

and putting for u its value, there results

$$\frac{dy}{du} = \frac{2\left(a+\sqrt{b-\frac{c}{x^2}}\right)^3}{\sqrt{b-\frac{c}{x^2}}};$$

multiplying these differential coefficients together, we have lastly

$$\frac{dy}{dx} = \frac{\frac{4c}{x^3}\left(a+\sqrt{b-\frac{c}{x^2}}\right)^3}{\sqrt{b-\frac{c}{x^2}}}.$$

We might take also as an example

$$y = (a+\sqrt{x})^3,$$

and we should find

$$\frac{dy}{dx} = \frac{3(a+\sqrt{x})^2}{2\sqrt{x}}.$$

On successive differentiation.

30. Let y be a function of x; if this be differentiated, we shall find a result of the form pdx, p being a quantity which may involve x; if p do involve x, we shall be able to differentiate p also, and so obtain a result represented by qdx; pro-

ceeding in the same manner with respect to q, the result will be of the form rdx, and so on: pdx, qdx, rdx, &c. are the successive differentials of y.

For example, if $y=ax^3$, we shall find $dy=3ax^2dx$, and therefore $p=3ax^2$; differentiating anew, we have $dp=6axdx$, and therefore $q=6ax$. Again differentiating, $dq=6adx$, whence $r=6a$; and here the differentiation must stop, $6a$ being a constant.

The equations

$$dy=pdx \text{ give, dividing by } dx,\ \frac{dy}{dx}=p,$$

$$dp=qdx \ldots\ldots\ldots\ldots \frac{dp}{dx}=q,$$

$$dq=rdx \ldots\ldots\ldots\ldots \frac{dq}{dx}=r;$$

&c. &c.

q being obtained by two successive differentiations, and by dividing each time by dx, we will represent the operation by $\frac{d^2y}{dx^2}$, and we shall have $\frac{d^2y}{dx^2}=q$; in like manner by differentiating anew, and dividing by dx, we have $\frac{d^3y}{dx^3}=r$; and so on.

dy is the first differential of y;

d^2y is the second differential;

d^3y is the third differential;

and so on.

Maclaurin's theorem.

31. Let y be a function of x, arrange it according to the powers of x, and suppose

$$y=A+Bx+Cx^2+Dx^3+Ex^4+\&c. \ldots \quad (16);$$

then differentiating and dividing by dx we shall find

$$\frac{dy}{dx}=B+2Cx+\quad 3Dx^2+\quad 4Ex^3+\text{ \&c.}$$

$$\frac{d^2y}{dx^2}=\qquad 2C+3.2.Dx+4.3.Ex^2+\text{ \&c.}$$

$$\frac{d^3y}{dx^3}=\qquad\qquad 3.2.D\ +4.3.2.Ex+\text{ \&c.}$$

&c.

Represent by (y) the value assumed by y when $x=0$,

$$\text{by}\left(\frac{dy}{dx}\right)\ \ldots\ldots\ldots\ldots\ \frac{dy}{dx}\ \ldots\ldots,$$

$$\text{by}\left(\frac{d^2y}{dx^2}\right)\ \ldots\ldots\ldots\ldots\ \frac{d^2y}{dx^2}\ \ldots\ldots,$$

and so on;

the preceding equations will give us

$$(y)=A,\left(\frac{dy}{dx}\right)=B,\left(\frac{d^2y}{dx^2}\right)=2C,\left(\frac{d^3y}{dx^3}\right)=3.\,2.\,D,\text{ \&c.}$$

whence we find

$$A=(y),\,B=\left(\frac{dy}{dx}\right),\ C=\frac{1}{2}\left(\frac{d^2y}{dx^2}\right),\ D=\frac{1}{3.2}\left(\frac{d^3y}{dx^3}\right),\text{ \&c.}$$

and substituting these values in the equation (16), it will become

$$y=(y)+\left(\frac{dy}{dx}\right)x+\frac{1}{2}\left(\frac{d^2y}{dx^2}\right)x^2+\frac{1}{2.3}\left(\frac{d^3y}{dx^3}\right)x^3+\text{ \&c. (17)},$$

which is Maclaurin's theorem.

32. As a first application let us take $y=\dfrac{1}{a+x}$;

differentiating, we find

$$dy=\frac{(a+x)\,d.\,1-1.d\,(a+x)}{(a+x)^2}=-\frac{dx}{(a+x)^2},$$

whence we deduce $\dfrac{dy}{dx}=-\dfrac{1}{(a+x)^2}$;

and differentiating anew, we shall get successively

$$\frac{d^2y}{dx^2}=\frac{2(a+x)}{(a+x)^4}=\frac{2}{(a+x)^3}.$$

$$\frac{d^3y}{dx^3} = -\frac{2.3(a+x)^2}{(a+x)^6} = -\frac{2.3}{(a+x)^4}.$$

&c.

Making then $x=0$ in the values of $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}$, we find

$$(y)=\frac{1}{a}, \left(\frac{dy}{dx}\right) = -\frac{1}{a^2}, \left(\frac{d^2y}{dx^2}\right) = \frac{2}{a^3}, \frac{d^3y}{dx^3} = -\frac{2.3}{a^4};$$

and substituting these values and that of y in formula (17), we shall obtain

$$\frac{1}{a+x} = \frac{1}{a} - \frac{x}{a^2} + \frac{x^2}{a^3} - \frac{x^3}{a^4} + \text{\&c.}$$

33. For a second application, take $y = \sqrt{a^2+bx}$; we have then

$$y = (a^2+bx)^{\frac{1}{2}},$$

$$\frac{dy}{dx} = \tfrac{1}{2}(a^2+bx)^{-\frac{1}{2}}.b = \frac{1}{2}.\frac{b}{\sqrt{a^2+bx}},$$

$$\frac{d^2y}{dx^2} = -\tfrac{1}{2}.\tfrac{1}{2}(a^2+bx)^{-\frac{3}{2}}b^2 = -\frac{\frac{1}{2}.\frac{1}{2}.b^2}{\sqrt{(a^2+bx)^3}},$$

$$\frac{d^3y}{dx^3} = \tfrac{1}{2}.\tfrac{1}{2}.\tfrac{3}{2}(a^2+bx)^{-\frac{5}{2}}b^3 = \frac{\frac{1}{2}.\frac{1}{2}.\frac{3}{2}.b^3}{\sqrt{(a^2+bx)^5}};$$

making $x=0$, these values will become

$$(y) = (a^2)^{\frac{1}{2}} = a, \left(\frac{dy}{dx}\right) = \tfrac{1}{2}.\frac{b}{a}, \left(\frac{d^2y}{dx^2}\right) = -\tfrac{1}{2}.\tfrac{1}{2}.\frac{b^2}{a^3}, \left(\frac{d^3y}{dx^3}\right) = \frac{\frac{1}{2}.\frac{1}{2}.\frac{3}{2}b^3}{a^5};$$

and substituting these values of $(y), \left(\frac{dy}{dx}\right)$ &c. in formula (17), we shall find

$$\sqrt{a^2+bx} = a + \frac{bx}{2a} - \frac{b^2x^2}{8a^3} + \frac{b^3x^3}{16a^5} - \text{\&c.}$$

34. As a third example, take $y = (a+x)^m$; differentiating we shall find

$$\frac{dy}{dx} = m\,(a+x)^{m-1},$$

$$\frac{d^2y}{dx^2} = m\,(m-1)\,(a+x)^{m-2},$$

$$\frac{d^3y}{dx^3} = m\,(m-1)\,(m-2)\,(a+x)^{m-3};$$

making $x=0$, the value of y is reduced to a^m, whence $(y) = a^m$; and the differential coefficients $\frac{dy}{dx}, \frac{d^2y}{dx^2}$, &c. give us

$$\left(\frac{dy}{dx}\right) = ma^{m-1}, \left(\frac{d^2y}{dx^2}\right) = m\,(m-1)\,a^{m-2}, \frac{d^3y}{dx^3} = \;\ldots\ldots$$

$m(m-1)\,(m-2)\,a^{m-3}$, which values of (y), $\left(\frac{dy}{dx}\right)$, $\left(\frac{d^2y}{dx^2}\right)$, &c. being substituted in formula (17), we find

$$(a+x)^m = a^m + ma^{m-1}x + m\frac{(m-1)}{2}\,a^{m-2}x^2$$
$$+ m\,\frac{(m-1)}{2}.\frac{(m-2)}{3}\,a^{m-3}x^3 + \text{\&c.}$$

On the differentiation of transcendental quantities.

35. Transcendental quantities are such as are affected by variable indices, logarithms, sines, cosines, &c.

36. Let a^x be the quantity first proposed to be differentiated; put $y = a^x$, change x into $x+h$, and y into y', when the equation will become

$$y' = a^{x+h}, \text{ or } y' = a^x a^h,$$

and this expression we must develope according to powers of h. In order, therefore, that a^h may be developable by the binomial theorem, we will put $a = 1 + b$, and consequently a^h will become

$$(1+b)^h = 1 + h.\frac{b}{1} + h\,(h-1).\frac{b^2}{1.2} + h.(h-1)\,(h-2)$$
$$\frac{b^3}{2.3} + \text{\&c.} \;\ldots\ldots\; (18);$$

which we might arrange in respect of h; but without performing this operation, since we want only those terms multiplied by the first power of h, we will observe that if in a product such as $h\ (h-1)\ (h-2)\ (h-3)$ &c., the part $(h-1)$ $(h-2)$ &c. is composed of n factors, its development, according to the theory of equations, will be of the form $h^n+\text{A}h^{n-1}+\text{B}h^{n-2}\ldots+\text{M}h+\text{N}$, and the term N will be formed of the continued product of the second parts -1, -2, -3, &c. of the binomials $h-1$, $h-2$, $h-3$, &c. But since $h\ (h-1)$ $(h-2)\ (h-3)$ &c. $=h\ (h^n+\text{A}h^{n-1}\ldots\ \text{M}h+\text{N})$, it is evident that the term containing the first power of h in that product will be Nh, or from what has preceded, $h\ (-1\times-2\times-3)$ &c. whence we may conclude that to find in the development (18) the terms involving the first power of h, we must, in the more complicated terms of that series, beginning, for instance, with the third, form the several co-efficients of h in the manner following; the continued product of the numbers subtracted from h in the several terms must be multiplied in the third term by $\frac{b^2}{1.2}$, in the fourth term by $\frac{b^3}{2.3}$, and so on; whence it follows that

$$a^h=1+(b-\frac{b^2}{2}+\frac{b^3}{3}-\text{ \&c.})\ h+\text{ terms involving } h^2,\ h^3,\text{ \&c.}$$

Representing $(b-\frac{b^2}{2}+\frac{b^3}{3}-$, &c.) by A, we have

$$a^h=1+\text{A}h+\text{ terms involving } h^2,\ h^3,\text{ \&c.};$$

and substituting this value in the equation $y'=a^xa^h$, that equation will become

$$y'=a^x+\text{A}a^xh.+\text{ terms involving } h^2,\ h^3,\text{ \&c.}$$

If we subtract the primitive equation $y=a^x$, there will remain

$$y'-y=\text{A}a^xh+\text{ terms involving } h^2,\ h^3,\text{ \&c.}$$

and taking the limit, $\frac{dy}{dx}=\text{A}a^x$,

or, replacing the value of y,

$$\frac{d.a^x}{dx}=\text{A}.a^x\ \ldots\ (19).$$

The constant A depends on a, for if in the equation

$$A = \left(b - \frac{b^2}{2} + \frac{b^3}{3} - \&c.\right),$$

we put for b its value $(a-1)$, we shall find

$$A = (a-1) - \frac{(a-1)^2}{2} + \frac{(a-1)^3}{3} - \&c. \; . \; . \; . \; . \; . \; . \; . \quad (20).$$

37. To determine the value of A, let us investigate, by Maclaurin's theorem, the development of a^x; we have then

$$y = a^x$$

$$\frac{dy}{dx} = Aa^x,$$

$$\frac{d^2y}{dx^2} = \frac{Ad.a^x}{dx} = \frac{A^2a^x dx}{dx} = A^2a^x,$$

$$\frac{d^3y}{dx^3} \; . \; . \; . \; . \; . \; . \; . \; . \; = A^3a^x. \; \&c.$$

and making $x = o$, we shall find

$$(y) = a^0 = 1, \left(\frac{dy}{dx}\right) = A, \frac{d^2y}{dx^2} = A^2, \left(\frac{d^3y}{dx^3}\right) = A^3, \; \&c.$$

Substituting these values in equation (17) we have

$$a^x = 1 + \frac{Ax}{1} + \frac{A^2x^2}{1.2} + \frac{A^3x^3}{1.2.3} + \&c.;$$

making $x = \frac{1}{A}$, this equation will become

$$a^{\frac{1}{A}} = 1 + 1 + \frac{1}{1.2} + \frac{1}{1.2.3} + \&c.;$$

and representing by e the second side of this equation, it will be changed into $a^{\frac{1}{A}} = e$, whence we find $a = e^A$; and taking the logarithms, we have

$$\log a = \log e^A = A \log e;$$

therefore

$$A = \frac{\log a}{\log e} \; . \; . \; . \; . \; . \; . \quad (21).$$

The number e, whose value is given by the equation $e=1+1+\frac{1}{1.2}+\frac{1}{1.2.3}+$ &c., is the base which Napier selected for calculating his tables of logarithms.

The series $1+1+\frac{1}{1.2}+\frac{1}{1.2.3}+$ &c. being sufficiently convergent, we may take the first ten terms as an approximation, when we shall find the value of e to be about 2,7182818. If we represent by $\mathrm{L}a$ the logarithm of a in the Napierian system, we shall have then $a=(2{,}7182818)^{\mathrm{L}a}$, or more simply $a=e^{\mathrm{L}a}$; and therefore $\log a=\log e^{\mathrm{L}a}=\mathrm{L}a\log e$; whence we shall find $\frac{\log a}{\log e}=\mathrm{L}a$, which reduces equation (21) to $\mathrm{A}=\mathrm{L}a$, and consequently equation (19) gives

$$d.a^x=a^x dx\mathrm{L}a \quad . \; . \; . \; . \quad (22).$$

Logarithmic differentials.

38. Let x be the logarithm of y in the system whose base is a; then we have $y=a^x$, and therefore (art. 36) $dy=\mathrm{A}a^x dx$; whence we find

$$dx=\frac{dy}{\mathrm{A}a^x}=\frac{dy}{\frac{\log a}{\log e}.a^x}=\frac{dy}{a^x}.\frac{\log e}{\log a};$$

and since $a^x=y$ and $x=\log y$, the preceding equation becomes

$$d.\log y=\frac{dy}{y}.\frac{\log e}{\log a}.$$

When we take the logarithms in the Napierian system, $\frac{\log e}{\log a}=\frac{\log e}{\log e}=1$, and therefore in that case $d.\log y=\frac{dy}{y}$.

The differentials of sines, cosines, and other trigonometrical lines, or the differentials of circular functions.

39. *The arc is greater than the sine, and less than the tangent.*

To prove this, let AB, fig. 2, be an arc, which has BE for its sine, and DA for its tangent, and take the arc AB′ equal to the arc AB. Then the chord BB′ being a straight line, BB′ is less than the arc AB; and therefore the straight line BE, which is the half of the chord BB′, is less than the arc BA, the half of arc BAB′; whence it follows that the sine is less than the arc.

To prove that the tangent is greater than the arc, we have

Area of triangle DD′C $>$ *area of sector BAB′C*;

or, putting for these areas their geometrical values,

$$\mathrm{DD}' \times \tfrac{1}{2}\mathrm{AC} > \mathrm{BAB}' \times \tfrac{1}{2}\mathrm{AC};$$

suppressing the common factor $\frac{1}{2}$AC, there remains

$$\mathrm{DD}' > \mathrm{BAB}',$$

and taking the halves, we have

$$\mathrm{DA} > \text{arc BA}.$$

40. It follows from this, that the limit of the ratio of the sine to the arc is unity; for since, when the arc h, represented by AB, becomes nothing, the sine coincides with the tangent; much more does the sine coincide with the arc, which lies between the tangent and sine; and, consequently, we have, in the case of the limit, $\frac{\sin h}{\text{arc } h}$, or rather, $\frac{\sin h}{h}=1$.

41. To find the differential of the sine whose arc is x, suppose that the arc receives an increment h; then we know, by trigonometry, that

$$\sin(x+h)=\sin x \,.\, \cos h+\cos x \,.\, \sin h \quad \ldots\ldots \quad (23).$$

Subtracting from this function its primitive, and dividing by the increment h of the variable, we shall have

$$\frac{\sin(x+h)-\sin x}{h}=\frac{\sin x \,.\, \cos h+\sin h \,.\, \cos x-\sin x}{h};$$

and collecting together the terms multiplied by $\sin x$ on the second side, we shall find

$$\frac{\sin(x+h)-\sin x}{h}=\frac{\sin x\,(\cos h-1)}{h}+\frac{\sin h \,.\, \cos x}{h} \quad \ldots\ldots \quad (24).$$

When h becomes 0, $\cos h-1$ becomes also 0, and $\dfrac{\cos h-1}{h}$ is reduced to the form $\dfrac{0}{0}$; it is necessary, therefore, to put that term under some other form, and, for this purpose, the equation $\cos^2 h+\sin^2 h=1$, gives us $\cos^2 h-1=-\sin^2 h$, or $(\cos h-1)(\cos h+1)=-\sin^2 h$; from which we get

$$\cos h-1=-\frac{\sin^2 h}{\cos h+1};$$

and substituting this value in equation (24), it becomes

$$\frac{\sin(x+h)-\sin x}{h}=-\sin x.\frac{\sin h}{\cos h+1}+\cos x.\frac{\sin h}{h}\ \ldots\ (25).$$

When $h=0$, $\dfrac{\sin h}{h}=1$, $\dfrac{\sin h}{\cos h+1}=\dfrac{0}{2}=0$;

and therefore equation (25) is reduced to $\dfrac{d.\sin x}{dx}=\cos x$: whence we deduce $d.\sin x=\cos x.dx$.

42. In this demonstration, the radius of the tables has been supposed unity. If we wish to have the differential of a sine whose radius is a, instead of employing the equation (23), we must make use of this,

$$\sin(x+h)=\frac{\sin x.\cos h+\sin h.\cos x}{a};$$

and therefore, in the preceding result, it will be necessary to introduce the constant a, which will give $d.\sin x=\dfrac{dx.\cos x}{a}$, for the differential of the sine of an arc whose radius is a.

43. We might arrive at the differential of $\sin x$ by geometrical considerations; for, let AB, fig. 3, be the arc x, BM the arc h; then the perpendicular BP will be the $\sin x$, and the perpendicular MQ the $\sin(x+h)$. This being supposed, the more the arc $\text{BM}=h$ is diminished, the more does the angle MBC approximate to a right angle; and consequently, in the case of the limit, we may consider MBC as a right angle, and Fig. 3.

the triangle MBD will then become similar to the triangle BCP; since in that case the triangles have their sides perpendiculars. From this it follows, that we have the proportion

$$BC : CP :: BM : MD,$$

or,

$$r : \cos x :: BM : \sin(x+h) - \sin x;$$

and therefore

$$\frac{\sin(x+h) - \sin x}{BM} = \frac{\cos x}{r};$$

taking the limit, and observing that, in this case, the chord BM may be replaced by the arc $BM = h$, the above equation becomes

$$\frac{d.\sin x}{dx} = \frac{\cos x}{r};$$

and taking the radius equal to unity,

$$d.\sin x = dx \cos x.$$

44. To find the differential of $\cos x$, the equation $\sin^2 x + \cos^2 x = 1$, or rather, $(\sin x)^2 + (\cos x)^2 = 1$, being differentiated, gives $2 \sin x d. \sin x + 2 \cos x d. \cos x = 0$; whence we find

$$d. \cos x = -\frac{\sin x d. \sin x}{\cos x};$$

and putting for $d. \sin x$ its value, $dx. \cos x$, art. 41, and reducing, we have $d. \cos x = -dx. \sin x$.

45. We obtain the differential of tangent x, by considering that $\tan x = \frac{\sin x}{\cos x}$; which equation being differentiated by art. (16), we find

$$d. \tan x = \frac{\cos x d. \sin x - \sin x d. \cos x}{\cos^2 x};$$

and putting the values of $d.\sin x$ and $d. \cos x$, we shall have

$$d.\tan x = \left(\frac{\cos^2 x + \sin^2 x}{\cos^2 x}\right) dx,$$

and therefore, since $\cos^2 x + \sin^2 x = 1$,

$$d.\tan x = \frac{dx}{\cos^2 x}.$$

46. We know, by trigonometry, that the radius is a mean proportional between the tangent and the cotangent, and between the cosine and the secant, which gives

$$\cot x = \frac{1}{\operatorname{tang} x}, \quad \sec x = \frac{1}{\cos x}.$$

By differentiating the first of these equations (art. 16), we find

$$d.\cot x = -\frac{d.\operatorname{tang} x}{\operatorname{tang}^2 x} = -\frac{dx}{\cos^2 x.\ \operatorname{tang}^2 x} = -\frac{dx}{\sin^2 x};$$

for, from the equation $\frac{\sin}{\cos} = \operatorname{tang}$, we deduce $\cos.\ \operatorname{tang} = \sin$.

47. The equation $\sec x = \frac{1}{\cos x}$, being differentiated, gives

$$d.\sec x = -\frac{d.\cos x}{\cos^2 x} = \frac{\sin x dx}{\cos^2 x} = \frac{\sin x}{\cos x}.\frac{dx}{\cos x} = \operatorname{tang} x.\sec x dx.$$

48. We may determine in the same manner the differential of the cosecant; for $\operatorname{cosec} x = \frac{1}{\sin x}$; which, being differentiated, we have

$$d.\operatorname{cosec} x = -\frac{\cos x dx}{\sin^2 x} = -\frac{\cos x}{\sin x}.\frac{dx}{\sin x} = -\frac{1}{\tan x}.\operatorname{cosec} x dx$$
$$= -\cot x \operatorname{cosec} x dx.$$

49. In respect of the versin, by differentiating the equation $\operatorname{versin} x + \cos x = 1$, we find $d.\operatorname{versin} x = d(1 - \cos x)$, and by performing the differentiation,

$$d.\operatorname{versin} x = \sin x dx.$$

On the differentiation of certain complicated transcendental functions.

50. The preceding principles will serve for the differentiating every expression affected with transcendental quantities.

Let $y = a^{b^x}$ be the function; making $b^x = u$, we shall have $y = a^u$, and differentiating by art. 37, we shall find

$$\frac{dy}{du} = a^u \mathrm{L}a = a^{b^x} \mathrm{L}a, \quad \frac{du}{dx} = b^x \mathrm{L}b;$$

therefore (art. 26),

$$\frac{dy}{du} \cdot \frac{du}{dx}, \text{ or } \frac{dy}{dx} = a^{b^x} b^x \mathrm{L}a \, \mathrm{L}b.$$

51. Let also $y = z^v$; taking the logarithms, we have $\log y = v \log z$; and therefore, $d . \log y = vd . \log z + \log z dv$: putting for the logarithmic differentials their values (art. 38), we shall find

$$\frac{dy}{y} = v\frac{dz}{z} + \log z dv.$$

and consequently,

$$dy = y\left(\frac{vdz}{z} + \log z dv\right), \text{ or, } dy = z^v\left(\frac{vdz}{z} + \log z dv\right).$$

By means of this differential, we shall easily find that of $y = z^{t^u}$: for, let $t^u = v$, then the equation is reduced to $y = z^v$, and the equations $y = z^v$, $v = t^u$, being of the same form with the equation whose differential we have just found, will give

$$dy = z^v\left(v\frac{dz}{z} + \log z dv\right)$$

$$dv = t^u\left(u\frac{dt}{t} + \log t du\right).$$

Substituting the values of v and dv in that of dy given by the last equation but one, we shall have

$$dy = z^{t^u}\left[t^u\frac{dz}{z} + \log z . t^u \left(u\frac{dt}{t} + \log t . du\right)\right]$$

$$= z^{t^u} . t^u \left(\frac{dz}{z} + u \log z\frac{dt}{t} + \log z . \log t . du\right).$$

Taylor's theorem.

52. Before proceeding farther, we will observe that in the differential calculus an expression such as $\frac{dy}{dx}$ signifies that a function y of one or more variables has been differentiated in respect of the variable x and divided by dx; if, for instance, we had $y = ax^2u^3z^4$, the expression $\frac{dy}{dx}$ would be found by considering u and z as constant, differentiating in respect of x, and dividing by dx, so that we shall have $\frac{dy}{dx} = 2axu^3z^4$. We should find in the same manner $\frac{dy}{dz} = 4ax^2z^3u^3$ and $\frac{dy}{du} = 3ax^2z^4u^2$. If we had $y = x^2 + z^2$, $\frac{dy}{dx}$ would be $2x$.

53. *If in a function* y *of* x, *the variable* x *is changed into* x+h, *we have the same differential coefficient when* x *is variable and* h *constant, as when* h *is variable and* x *constant.*

To demonstrate this, if in the equation $y = fx$, we put $x + h = x'$ in place of x, we shall have $y' = fx'$; the differential of fx' will be equal to some other function of x' represented by $\varphi x'$ and multiplied by dx, and consequently $dy' = \varphi x'dx'$, or putting for x' its value $x + h$, we shall have

$$dy' = \varphi(x+h)\, d\,(x+h)\,;$$

in which differential the only change arising from the hypothesis of x being variable and h constant is in the factor $d(x+h)$, which is then reduced to dx, so that we have in that case

$$dy' = \varphi\,(x+h)\,dx,$$

whence we find

$$\frac{dy'}{dx} = \phi\,(x+h)\quad .\ .\ .\ .\ (26).$$

If, on the contrary, we make h variable and x constant, the factor $d(x+h)$ is reduced to dh, and we have

$$dy' = \varphi(x+h)\,dh,$$

and consequently

$$\frac{dy'}{dh} = \varphi(x+h) \quad . \quad . \quad . \quad . \quad (27):$$

equating these two values of $\varphi(x+h)$, there results

$$\frac{dy'}{dx} = \frac{dy'}{dh}.$$

For example, if we had $y = ax^3$, by putting $x+h$ for x, we should find

$$\frac{dy'}{dx} = 3a(x+h)^2 = \frac{dy'}{dh},$$

and consequently

$$\frac{dy}{dx} = \frac{dy'}{dh}.$$

54. The equations (26) and (27) being differentiated in respect of $x+h$, give still the equal results

$$\frac{d^2y'}{dx} = \varphi'(x+h)\,d(x+h)$$

$$\frac{d^2y'}{dh} = \varphi'(x+h)\,d(x+h);$$

and making h constant in the first equation, and x constant in the second, we shall have

$$\frac{d^2y'}{dx} = \varphi'(x+h)\,dx, \quad \frac{d^2y'}{dh} = \varphi'(x+h)\,dh,$$

whence we shall deduce

$$\frac{d^2y'}{dx^2} = \frac{d^2y'}{dh^2}.$$

We may conclude by similar reasoning that $\frac{d^3y'}{dx^3} = \frac{d^3y'}{dh^3}$, $\frac{d^4y'}{dx^4} = \frac{d^4y'}{dh^4}$; and so on.

55. This being premised, let y' be a function of $x+h$, and suppose that, when this function is developed according to the powers of h, we have

$$y'=y+Ah+Bh^2+Ch^3+\&c. \dots\dots\dots (28);$$

A, B, C, &c., being unknown functions of x, now to be determined. For this purpose, differentiating in respect of h, and dividing by dh, we shall have

$$\frac{dy'}{dh}=A+2Bh+3Ch^2+\&c.$$

Differentiating again in respect of x, and dividing by dx, we shall have

$$\frac{dy'}{dx}=\frac{dy}{dx}+\frac{dA}{dx}h+\frac{dB}{dx}h^2+\&c.;$$

and the first sides of these equations being equal, by article 53, the second sides must be identical; whence, equating the coefficients of the same powers of h, we shall find

$$A=\frac{dy}{dx}, \quad B=\frac{dA}{2dx}, \quad C=\frac{dB}{3dx}, \quad D=\frac{dC}{4dx}, \ \&c.$$

Substituting the value of A, given by the first of these equations, in the second, we shall have $B=\frac{1}{1.2}\frac{d^2y}{dx^2}$; substituting this value in that of C, we shall have $C=\frac{1}{1.2.3}\frac{d^3y}{dx^3}$; and so on.

By means of these values of A, B, C, &c., the equation (28) will become

$$y'=y+\frac{dy}{dx}h+\frac{d^2y}{dx^2}\cdot\frac{h^2}{1.2}+\frac{d^3y}{dx^3}\,\frac{h^3}{1.2.3}+\&c.;$$

or, putting for y' its value,

$$f(x+h)=y+\frac{dy}{dx}h+\frac{d^2y}{dx^2}\,\frac{h^2}{1.2}+\frac{d^3y}{dx^3}\,\frac{h^3}{1.2.3}+\&c.,$$

which is Taylor's theorem.

Application of Taylor's theorem to the development of different functions in the form of a series.

56. Let $y' = \sqrt{x+h}$ be the function:
we have then

$$y=\sqrt{x}=x^{\frac{1}{2}},$$

and therefore

$$\frac{dy}{dx}=\frac{1}{2}x^{-\frac{1}{2}}=\frac{1}{2\sqrt{x}},$$

$$\frac{d^2y}{dx^2}=-\frac{1}{4}x^{-\frac{3}{2}}=-\frac{1}{4}\cdot\frac{1}{x^{\frac{3}{2}}},$$

$$\frac{d^3y}{dx^3}=\frac{3}{8}x^{-\frac{5}{2}}=\frac{3}{8}\frac{1}{x^{\frac{5}{2}}},\ \text{\&c.}$$

substituting in the formula we have

$$\sqrt{x+h}=\sqrt{x}+\frac{1}{2}\frac{h}{\sqrt{x}}-\frac{1}{8}\frac{h^2}{\sqrt{x^3}}+\frac{1}{16}\frac{h^3}{\sqrt{x^5}},\ \text{\&c.}$$

57. Let $y'=\sin(x+h)$, whence it follows that $y=\sin x$; and we may therefore form the successive differential coefficients thus;

$$\frac{dy}{dx}=\cos x;\ \frac{d^2y}{dx^2}=-\sin x;\ \frac{d^3y}{dx^3}=-\cos x;$$

$$\frac{d^4y}{dx^4}=\sin x;\ \frac{d^5y}{dx^5}=\cos x;\ \text{\&c.};$$

and substituting in Taylor's formula, we find

$$\sin(x+h)=\sin x+\cos x.\frac{h}{1}-\sin x\frac{h^2}{1.2}-\cos x.\frac{h^3}{1.2.3}$$

$$+\sin x.\frac{h^4}{1.2.3.4}+\cos x\frac{h^5}{2.3.4.5},\ \text{\&c.}$$

Making $x=0$, $\sin x$ will then be $=0$, $\cos x=1$, and the series will become

$$\sin h=h-\frac{h^3}{1.2.3}+\frac{h^5}{1.2.3.4.5}-\ \text{\&c.}$$

If we took $y' = \cos(x+h)$, we should find, proceeding as in the last example, that

$$\cos h = 1 - \frac{h^2}{2} + \frac{h^4}{2.3.4} - \&c.$$

58. Let us develope also $\log(x+h)$; then we have

$$y' = \log(x+h), \text{ and therefore } y = \log x;$$

$$dy = d.\log x = \frac{dx}{x}, \text{ and therefore } \frac{dy}{dx} = \frac{1}{x};$$

and we shall obtain then, by successive differentiations,

$$\frac{d^2y}{dx^2} = -\frac{1}{x^2}, \quad \frac{d^3y}{dx^3} = \frac{2}{x^3}; \ \&c.$$

substituting these values in the formula of Taylor, we have

$$\log(x+h) = \log x + \frac{h}{x} - \frac{h^2}{2x^2} + \frac{h^3}{3x^3}, \ \&c.$$

59. Had this formula been deduced from the principles of Algebra alone, and not by differentiation, we might easily, by means of it, find the differential of a logarithm [*note first.*] For the formula gives

$$\frac{\log(x+h) - \log x}{h} = \frac{1}{x} - \frac{h}{2x^2} + \&c.;$$

and taking the limit, we have

$$\frac{d.\log x}{dx} = \frac{1}{x}$$

$$d.\log x = \frac{dx}{x}.$$

Knowing the differential of a logarithm, it would be easy to find that of a^x; for by making $y = a^x$, and taking the logarithms in the Napierian system, we have

$$\mathrm{L}y = \mathrm{L}a^x = x\mathrm{L}a,$$

and differentiating,

$$\frac{dy}{y} = dx\mathrm{L}a,$$

whence we find

$$dy = ydx\mathrm{L}a = a^x dx\mathrm{L}a.$$

60. Maclaurin's theorem may be deduced from that of Taylor in the manner following; we have, by Taylor's theorem,

$$f(x+h) = fx + \frac{d.fx}{dx}h + \frac{d^2fx}{dx^2}\cdot\frac{h^2}{1.2} + \frac{d^3fx}{dx^3}\,\frac{h^3}{1.2.3} + \text{\&c.}$$

Representing by $(fx), \left(\frac{d.fx}{dx}\right)$, &c. the values of $fx, \frac{dfx}{dx}$, &c. when we make $x=0$; the formula of Taylor will become, when $x=0$,

$$fh = (fx) + \left(\frac{d.fx}{dx}\right)h + \left(\frac{d^2fx}{dx^2}\right)\frac{h^2}{1.2} + \text{\&c.};$$

in which equation, h enters into fh as x entered into fx, so that if we change h into x, fh will become fx; and since there no longer remain any traces of x, this change is allowable, it being of little consequence whether we substitute one letter or another for h: making therefore this change, we find

$$fx = (fx) + \left(\frac{d.fx}{dx}\right)x + \left(\frac{d^2fx}{dx^2}\right)\frac{x^2}{1.2} + \text{\&c.};$$

which is Maclaurin's theorem.

On the differentiation of equations of two variables.

61. Let

$$\mathrm{F}\,(x,y) = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \quad (29)$$

be an equation betwixt two variables.

Resolving this equation in respect of y, we shall find $y = \varphi x$, and supposing that we have substituted this value in the equation (29), it will become $F(x, \varphi x) = 0$, or for greater simplicity

$$fx = 0,$$

an equation identical with the former, and in which all the terms must vanish, whatever value we give to x. If, for instance, the equation rise only to the third degree, we may represent it by

$$Ax^3 + Bx^2 + Cx + D = 0,$$

and putting any value whatever for x, this must be always satisfied; wherefore putting $x + h$ for x, we shall have still

$$A(x+h)^3 + B(x+h)^2 + C(x+h) + D = 0,$$

that is to say, if we have $fx = 0$, whatever be the value of x, we shall have also $f(x+h) = 0$.

Subtracting from this equation the former one $fx = 0$, there will remain

$$f(x+h) - fx = 0,$$

and therefore

$$\frac{f(x+h) - fx}{h} = 0.$$

But

$$f(x+h) = fx + Ah + Bh^2 + \&c.$$

whence we deduce

$$\frac{f(x+h) - fx}{h} = A + Bh + \&c.$$

the first side of which equation being 0, we have

$$A + Bh + \&c. = 0, \text{ and taking the limit, } \frac{d.fx}{dx} = A = 0,$$

and consequently $d.fx = Adx = 0$, or by restoring y,

$$d.F(x, y) = Adx = 0.$$

This shows us that considering y as a function of x, if we differentiate the equation $F(x, y) = 0$, we may put the result equal to 0; which will serve to determine the value of the differential coefficient $\frac{dy}{dx}$, as we shall see in the following example. Let

$$F(x, y) = x^2 + 3ay - y^2 = 0 \ldots\ldots\ldots (30);$$

differentiating by the ordinary processes, and observing that, from the preceding demonstration, we may put the result $= 0$, we have

$$2xdx + 3ady - 2ydy = 0 \ldots\ldots (31);$$

from which equation we find

$$\frac{dy}{dx} = \frac{2x}{2y - 3a} \ldots\ldots\ldots\ldots\ldots (32).$$

62. If we compare the process which has given us this value with that which we have hitherto employed, we shall see, that, working according to the previous method, it would have been necessary, first, to put the equation (30) under the form $y = fx$, and consequently to resolve the equation in respect to y, in order to deduce then by differentiation the value of $\frac{dy}{dx}$. Following that course, we should find, first,

$$y = \frac{3a}{2} \pm \sqrt{\frac{9}{4}a^2 + x^2};$$

and then, by differentiation,

$$\frac{dy}{dx} = \pm \frac{x}{\sqrt{\frac{9}{4}a^2 + x^2}}.$$

This value of $\frac{dy}{dx}$ appears under a form different from that presented to us by equation (32); but putting in equation (32) the value of y, that equation will become

$$\frac{dy}{dx} = \pm \frac{2x}{2\sqrt{\frac{9}{4}a^2 + x^2}} = \pm \frac{x}{\sqrt{\frac{9}{4}a^2 + x^2}}.$$

as we have just found. The equation (31) is the first differential of equation (30).

To obtain the equation which gives the second differential coefficient, i. e. $\frac{d^2y}{dx^2}$, dividing the equation (31) by dx, and making $\frac{dy}{dx}=p$, that equation will become

$$2x+3ap-2yp=0\ ;$$

considering y and p as functions of x, we shall have, by differentiation,

$$2dx+3adp-2ydp-2pdy=0\ ;$$

and dividing by dx, and putting p in place of $\frac{dy}{dx}$, there results

$$2+3a\frac{dp}{dx}-2y\frac{dp}{dx}-2p^2=0,$$

from which we find

$$\frac{dp}{dx}=\frac{2p^2-2}{3a-2y}\ .\ .\ .\ .\ .\ (33).$$

But since $\frac{dy}{dx}=p$, we shall have $\frac{dp}{dx}=\frac{d^2y}{dx^2}$, putting which values in equation (33), and getting quit of the denominators, we shall obtain

$$d^2y(3a-2y)=2dy^2-2dx^2\ .\ .\ .\ .\ .\ .\ (34),$$

which will be the second differential of the equation (30).

To obtain the third differential, we must put $\frac{dp}{dx}=q$, when, having got quit of the denominator, equation (33) will become

$$3aq-2yq=2p^2-2\ ;$$

and this being differentiated, considering y, p, q, as functions of x, we shall find the third differential; and so on for the rest.

63. Instead of using the letters p, q, r, for the performing

these operations, we might arrive at the same result by differentiating the equation (31), and putting dy for the differential of y, d^2y for that of dy, d^3y for that of d^2y, and considering dx as constant; by which means we should find

$$2dx^2+3ad^2y-2dy^2-2yd^2y=0;$$

the same with equation (34).

64. We will now give the general expression for the differential of the equation $f(x,y)=0$; for which purpose, representing $f(x,y)$ by u, we shall have, by differentiating the function in respect of x, the term $\frac{du}{dx}dx$; and by differentiating in respect of y, the second term, $\frac{du}{dy}dy$; so that

$$d.f(x,y), \text{ or } du=\frac{du}{dx}dx+\frac{du}{dy}dy.$$

But if y is considered as a function of x, we shall have, by differentiation,

$$dy=\frac{dy}{dx}dy;$$

which value being substituted, we shall find

$$du=\frac{du}{dx}dx+\frac{du}{dy}\frac{dy}{dx}dx.$$

65. Recalling to mind the theorem demonstrated, article (26), we shall see that u being considered as a function of y, and y as a function of x, the product $\frac{du}{dy}\frac{dy}{dx}$ is no other than the differential of u, taken in respect of x, contained in y.

66. The total differential of a function of x and y being given by the equation

$$du=\frac{du}{dx}dx+\frac{du}{dy}dy,$$

the expressions $\frac{du}{dx}dx$, $\frac{du}{dy}dy$ have been called the partial differentials of u. In like manner, if u be a function of three independent variables x, y, z, we shall have

$$du = \frac{du}{dx}dx + \frac{du}{dy}dy + \frac{du}{dz}dz,$$

and the terms,

$$\frac{du}{dx}dx, \quad \frac{du}{dy}dy, \quad \frac{du}{dz}dz,$$

will be the partial differentials of u.

67. We have seen (art. 52), that an expression such as $\frac{dy}{dx}$ indicates that the function y has been differentiated in respect of x, and then divided by dx; whence it follows, that if we have an equation $\frac{dy}{dx} = A$, and therefore,

$$1 = \frac{A}{\frac{dy}{dx}},$$

we cannot, without demonstration, conclude from it that

$$1 = A\frac{dx}{dy};$$

for in this new equation the differentiation is no longer made in respect of x, but in respect of y; and we do not yet know whether on this new hypothesis of differentiation the result will be the same.

For the removing this difficulty, we have demonstrated (art. 26) that

$$\frac{dv}{dx} = \frac{dv}{dy}\frac{dy}{dx}.$$

If in this equation we make $v = x$, it becomes

$$1 = \frac{dx}{dy} \cdot \frac{dy}{dx},$$

whence we find

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}},$$

which shows us that the change in the hypothesis of differentiation agrees with the principles of Algebra.

68. We will show how it may be demonstrated directly, that on the new hypothesis which gives the sign of division to the fraction $\frac{dy}{dx}$, the following equation holds good:

$$\frac{dx}{dy}=\frac{1}{\frac{dy}{dx}}.$$

Let

$$\frac{y'-y}{x'-x}=A+Bh+Ch^2+\&c.$$

then

$$\frac{x'-x}{y'-y}=\frac{1}{A+Bh+Ch^2+\&c.};$$

and effecting the division, or developing by means of Maclaurin's theorem, we obtain

$$\frac{x'-x}{y'-y}=\frac{1}{A}-\frac{B}{A}h+\&c.$$

Taking the limit, we have

$$\frac{dx}{dy}=\frac{1}{A};$$

and since $\frac{dy}{dx}=A$, it follows that

$$\frac{dx}{dy}=\frac{1}{\frac{dy}{dx}}.$$

On the method of tangents.

69. We give this name to the method which affords us the differential expressions for the tangents, subtangents, normals, and subnormals of curves.

Let x and y be the coordinates of a point M (fig. 4) taken in a curve; increase the abscissa $AP=x$ by a quantity $PP'=h$, draw the ordinate P′M′, and through the points M, M′, pass the secant M′S. Then it is evident that the more PP′ is diminished, the more PS tends to coincide with the subtangent PT, until at last $PP=h$ becomes 0. PT therefore will be the limit to which PS tends

We must now investigate the analytical expression for PS, in order to take the limit; and for this purpose the similar triangles M′MQ and MSP give the proportion,

$$\mathrm{M'Q} : \mathrm{MQ} :: \mathrm{MP} : \mathrm{PS},$$

or

$$\mathrm{M'Q} : h :: y : \mathrm{PS};$$

and therefore,

$$\mathrm{PS} = \frac{hy}{\mathrm{M'Q}}.$$

To determine M′Q, we have

$$\mathrm{M'Q} = \mathrm{M'P'} - \mathrm{MP};$$

but

$$\mathrm{M'P'} = y' = f(x+h),$$

and therefore,

$$\mathrm{M'P'} = y + \frac{dy}{dx}h + \frac{d^2y}{dx^2}\cdot\frac{h^2}{1.2} + \&c.$$

On the other hand

$$\mathrm{MP} = y;$$

if therefore we subtract these equations one from the other, there results

$$\mathrm{M'P} - \mathrm{MP}, \text{ or } \mathrm{M'Q} = \frac{dy}{dx}h + \frac{d^2y}{dx^2}\frac{h^2}{1.2} + \&c.$$

Substituting this value in that of PS, we shall find

$$\mathrm{PS} = \frac{hy}{\frac{dy}{dx}h + \frac{d^2y}{dx^2}\frac{h^2}{1.2} + \&c.}$$

and dividing by h,

$$\mathrm{PS} = \frac{y}{\frac{dy}{dx} + \frac{d^2y}{dx^2}\cdot\frac{h}{1.2} + \&c.}$$

At the limit, $h=0$ and PS becomes PT, which gives us

$$\mathrm{PT} = \frac{y}{\frac{dy}{dx}}, \text{ or (art 67) } \mathrm{PT} = y\frac{dx}{dy},$$

or rather,

$$\mathrm{PT} = y'\frac{dx'}{dy'} = \text{the } \textit{subtangent},$$

representing by x', y', the cordinates of the point M.

70. If at the point M (fig. 5) we draw MN perpendicular to MT, PN will be the subnormal. For determining it we have Fig. 5.

$$\mathrm{PT} : \mathrm{PM} :: \mathrm{PM} : \mathrm{PN},$$

or

$$y'\frac{dx'}{dy} : y' :: y' : \mathrm{PN},$$

and therefore

$$\mathrm{PN} = y'.\frac{dy}{dx'} = \text{the } \textit{subnormal}.$$

In respect to the tangent and the normal, we have

$$\mathrm{MT} = \sqrt{\mathrm{PT}^2 + \mathrm{PM}^2},$$

$$\text{or } \textit{tangent} = \sqrt{y'^2\frac{dx'^2}{dy'^2} + y'^2} = y'\sqrt{\frac{dx'^2}{dy'^2} + 1};$$

$$\mathrm{MN} = \sqrt{\mathrm{PN}^2 + \mathrm{PM}^2}.$$

$$\text{or } \textit{normal} = \sqrt{y'^2\frac{dy'^2}{dx'^2} + y'^2} = y'\sqrt{\frac{dy'^2}{dx'^2} + 1}.$$

71. To find the equation to the tangent, let x' and y' be the coordinates of the point of contact M; the equation to the straight line MT, passing through the point M, may then be represented by

$$y - y' = \mathrm{A}\,(x - x'),$$

where A is the trigonometrical tangent of the angle MTP, and will therefore be expressed by $\frac{\mathrm{PM}}{\mathrm{PT}}$; for we have

$$\mathrm{PT} : \mathrm{PM} :: 1 : \text{tang. MTP} = \frac{\mathrm{PM}}{\mathrm{PT}};$$

therefore

$$\text{tan. MTP} = \frac{\mathrm{PM}}{\mathrm{PT}} = \frac{y'}{\textit{subtangent}} = \frac{y'}{y'\frac{dx'}{dy'}} = \frac{dy'}{dx'}$$

Substituting this value of A in the equation to the tangent. that equation becomes

$$y - y' = \frac{dy'}{dx'}(x - x'), \textit{ the equation to the tangent.}$$

The equation to the normal will therefore be

$$y - y' = -\frac{dx'}{dy'}(x - x').$$

Application of the preceding formulæ to some examples.

72. 1°. *To find the subtangent of the parabola.*

The equation of the parabola being $y^2 = px$, we shall find by differentiation,

$$2ydy = pdx,$$

and consequently

$$\frac{dy}{dx} = \frac{p}{2y}.$$

But x' and y' are the coordinates of the point of contact, and in order therefore to have the differential coefficient corresponding to that point we must accent x and y; when we shall have

$$\frac{dy'}{dx'} = \frac{p}{2y};$$

substituting this value in that of PT, we obtain

$$\text{PT} = \frac{2y'^2}{p},$$

and putting px' in place of y'^2, that equation becomes

PT or the *subtangent* $= 2x'$.

2°. *To find the subnormal of the ellipse.* The equation of the ellipse, referred to the centre, is

$$b^2x^2 + a^2y^2 = a^2b^2,$$

which being differentiated, gives

$$2b^2xdx + 2a^2ydy = 0;$$

whence we find

$$\frac{dy'}{dx'} = -\frac{b^2}{a^2}\frac{x'}{y'}.$$

and putting this value in that of the subnormal PN, we obtain

$$\text{PN or the } \textit{subnormal} = -\frac{b^2}{a^2}x'.$$

3°. *To find the expression for the tangent to the circle.* The equation to the circle is $x^2+y^2=r^2$, which being differentiated, we find for the point of contact x', y',

$$\frac{dy'}{dx'} = -\frac{x'}{y'}.$$

By means of this value we shall reduce the expression for MT, the tangent, to

$$\textit{tangent} = y'\sqrt{\frac{y'^2}{x'^2}+1} = y'\sqrt{\frac{x'^2+y'^2}{x'^2}} = y'\sqrt{\frac{r^2}{x'^2}} = \frac{ry'}{x'}.$$

Asymptotes to curves.

73. The expression for AT (fig. 6), the distance of the vertex of the curve from the point T in the tangent, is readily deduced from the equation to the tangent; for if the vertex A of the curve be taken for the origin of the coordinates, the straight line AT will be the distance of that vertex from the point at which the ordinate PM becomes 0. Fig. 6.

Now the equation to the tangent is $y-y'=\frac{dy'}{dx'}(x-x')$, and it will therefore be sufficient to make $y=0$, in that equation, in order that the value of x, then deduced, may be that of AT; we obtain, in this manner,

$$\text{AT} = x' - y'\frac{dx'}{dy'};$$

which will be the distance of the origin from the point in which the tangent cuts the axis of x.

To determine the distance of the origin from the point in which the tangent cuts the axis of y, we must calculate the value of AB; and AB being the ordinate y, corresponding to $x=0$ in the equation to the tangent, we shall have on that hypothesis

$$\text{AB} = y' - \frac{dy'}{dx'}x'.$$

Suppose now that x becoming infinite, the values of AT and AB continue finite; we may conclude then that the straight line TL meets the curve only at an infinite distance; and consequently TL will be an asymptote to the curve.

74. Let us take, for example, the equation $y^2=mx+nx^2$; we shall find from this equation

$$\frac{dy'}{dx'}=\frac{m+2nx'}{2y'};$$

and therefore

$$\mathrm{AT}=x'-\frac{2y'^2}{m+2nx'}=\frac{mx'+2nx'^2-2y'^2}{m+2nx'}=-\frac{mx'}{m+2nx'},$$

$$\mathrm{AB}=y'-\frac{mx'+2nx'^2}{2y'}=\frac{2y'^2-mx'-2nx'^2}{2y'};$$

putting for y' its value, and reducing, we find

$$\mathrm{AT}=-\frac{mx'}{m+2nx'},\ \mathrm{AB}=\frac{mx'}{2\sqrt{mx'+nx'^2}};$$

and dividing each term in these fractions by x, we obtain

$$\mathrm{AT}=-\frac{m}{\frac{m}{x'}+2n},\ \mathrm{AB}=\frac{m}{2\sqrt{\frac{m}{x'}+n}}.$$

When x' is made infinite, these values become

$$\mathrm{AT}=-\frac{m}{2n},\ \mathrm{AB}=\frac{m}{2\sqrt{n}};$$

and therefore the curve will be susceptible of asymptotes, provided only that n be not either negative or 0; for if n be negative, in which case the equation belongs to an ellipse, the value of AB, given by the second of equations (35), becomes imaginary. The same equation is that of a parabola when n is 0; but on this hypothesis the equations show us that AT and AB become infinite, which proves that the parabola cannot have asymptotes.

On the equation of the tangent plane to a curve surface, and the equation of the normal to that surface.

Fig. 8. 75. Let $f(x, y, z)=0$ be the equation to a curve surface, and . . . . . . $Ax+By+Cz+D=0$ that to a plane. If the point M where the contact takes place have for its coordinates x', y', z', these coordinates must satisfy the equation of the plane, and we shall have consequently

$$Ax'+By'+Cz'+D=0.$$

Eliminating D betwixt this equation and the preceding one, we shall find for the equation of the plane made to pass through the point x', y', z',

$$A(x-x')+B(y-y')+C(z-z')=0 \ldots (36).$$

Draw through the point of contact x', y', z', a plane parallel to the plane of x, z; then this plane will cut the surface in a curve MC, and the tangent plane in a straight line ML (fig. 8), which must be a tangent to the curve MC, or otherwise the tangent plane would cut the curve surface.

The equation to the straight line ML may be deduced from equation (36); for the line ML being the section of the tangent plane made by a plane parallel to the plane of x, z, has at every point equal values for y; and since the point M is in that line, we have $y=y'$ or $y-y'=0$, which reduces the equation (36) to

$$A(x-x')+C(z-z')=0.$$

This equation will therefore express the relation that exists between the coordinates x and z of any point whatever, taken in the straight line ML, and consequently will be the equation to that line: it may be written thus:

$$z-z'=-\frac{A}{C}(x-x') \ldots\ldots (37).$$

The equation of the curve MC will be obtained in like manner, by considering y as constant, in the equation $f(x, y, z)=0$, to the curve surface.

In order to express the further condition of the straight line ML, being a tangent to the curve MC, we must (art. 71) have the coefficient of $x-x'$, in the equation (37), equal to the value of $\frac{dz'}{dx'}$, derived from the equation of the curve MC.

But the equation to that curve is the equation to the surface, considering y as constant; and it will therefore be sufficient to differentiate the equation to the surface, and derive from it $\frac{dz}{dx}$; for, according to art. 52, the notation $\frac{dz}{dx}$ supposes that we have considered y as a constant in the differentiation.

It follows, thence, that accenting z and x after the operation, we have for the condition of ML, being a tangent to MC,

$$-\frac{A}{C}=\frac{dz'}{dx'}, \text{ or } A=-C\frac{dz'}{dx'} \ldots\ldots (38).$$

If, again, we draw through the point M a plane, parallel to the plane of z, y, that plane will cut the surface in a curve MD, and the tangent plane in a straight line MN; and it may be demonstrated, as before, that the line

MN must be a tangent to the curve of intersection MD, and that for every point in the line MN, the abscissæ being equal, we must have $x - x' = 0$; which reduces the equation (36) to

$$B(y - y') + C(z - z') = 0;$$

whence we shall get

$$z - z' = -\frac{B}{C}(y - y').$$

This being the equation of the straight line MN, we shall express the condition of that line being a tangent to the curve MD, by equating the coefficient of $y - y'$ to the differential coefficient, $\frac{dz'}{dy'}$, deduced from the equation to the surface, when we shall have

$$-\frac{B}{C} = \frac{dz'}{dy'};$$

and consequently

$$B = -C\frac{dz'}{dy'} \dots\dots (39).$$

Substituting, in equation (36), the values of A and B, given by the equations (38) and (39), we shall find

$$-C\frac{dz'}{dx'}(x - x') - C\frac{dz'}{dy'}(y - y') + C(z - z') = 0;$$

whence we deduce for the equation of the tangent plane at the point x', y' z',

$$z - z' = \frac{dz'}{dx'}(x - x') + \frac{dz'}{dy'}(y - y') \dots\dots (40).$$

76. Let us find, for example, the equation of the tangent plane to a sphere. The coordinates of the centre being a, b, c, the sphere will have for its equation

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2;$$

and we find, by differentiating,

$$(x - a)dx + (y - b)dy + (z - c)dz = 0;$$

whence we deduce, according to the notation agreed on (art. 52),

$$\frac{dz}{dx} = \frac{a - x}{z - c}, \quad \frac{dz}{dy} = \frac{b - y}{z - c};$$

and the equation of the tangent plane to a sphere will therefore be at the point whose coordinates are x', y', z',

$$z - z' = \frac{a - x'}{z' - c}(x - x') + \frac{b - y'}{z' - c}(y - y').$$

77. If the plane passed through the extremity of the vertical diameter, we should have

$$x'=a,\ y'=b,\ z'=c+r\,;$$

and these values would reduce the equation of the tangent plane to $z=c+r$, the equation of a plane parallel to the plane of x, y.

78. The equations of the normal at the point x', y', z', may be readily deduced from the equation of the tangent plane: for, from the principles of analytical geometry (see my *Theory of Curves of the second order*, page 278), we know, that if we have the equations

$$\mathrm{A}x+\mathrm{B}y+\mathrm{C}z+\mathrm{D}=0\ \ldots\ldots\ (41),$$

$$\left.\begin{array}{l} x=az+\alpha \\ y=bz+\beta \end{array}\right\}\ \ldots\ldots\ldots\ (42),$$

the first being that of a plane, and the other two those of a straight line; the conditions necessary, in order that the line may be perpendicular to the plane, are that

$$\mathrm{A}=a\mathrm{C},\quad \mathrm{B}=b\mathrm{C}.$$

If, now, bringing over all the terms of equation (40) to the first side, we compare it with equation (41), we shall find, by equating the coefficients of x, y, z, in the two

$$\mathrm{A}=-\frac{dz'}{dx'},\ \ \mathrm{B}=-\frac{dz'}{dy'},\ \mathrm{C}=1\,:$$

and therefore

$$a=-\frac{dz'}{dx'},\ \ b=-\frac{dz'}{dy'}\,;$$

which values being substituted in the equation (42), we shall obtain

$$x=-\frac{dz'}{dx'}z+\alpha,\ \ y=-\frac{dz'}{dy'}z+\beta.$$

These equations, belonging to any point in the line, must hold for the point x', y', z', so that we shall have again

$$x'=-\frac{dz'}{dx'}z'+\alpha,\ \ y=-\frac{dz'}{dy'}z'+\beta\,;$$

and eliminating α and β betwixt these four last equations, we shall find, for the equations of the normal at the point x', y', z',

$$x-x'=-\frac{dz'}{dx'}(z-z'),\ \ y-y'=-\frac{dz'}{dy'}(z-z').$$

On functions which, for a particular value of the variable, become $\frac{0}{0}$.

79. When a fraction such as $\frac{\mathrm{F}x}{\phi x}$ becomes $\frac{0}{0}$, by substituting in it a particular value of x, which we will represent by a, it is a mark that the two terms of that fraction have $x-a$, or generally $(x-a)^n$ for a common factor; and if we can get quit of this, we shall have the true value of the fraction.

Suppose, therefore, that $x-a$ is m times a factor in $\mathrm{F}x$, and n times a factor in ϕx (admitting that, if the case require it, m and n may be assumed equal to unity or to zero), we may write then

$$\mathrm{F}x=\mathrm{P}(x-a)^m,\quad \phi x=\mathrm{Q}(x-a)^n,$$

and therefore

$$\frac{\mathrm{F}x}{\phi x}=\frac{\mathrm{P}}{\mathrm{Q}}(x-a)^{m-n}\ \ldots\ldots\ldots\ (43).$$

By differentiating, we find

$$\frac{d.\mathrm{F}x}{dx}=m\mathrm{P}(x-a)^{m-1}+\frac{d\mathrm{P}}{dx}(x-a)^m;$$

where, it will be observed, that the value of $\frac{d.\mathrm{F}x}{dx}$ consists of two terms, one of which contains a power of $(x-a)$, less by unity than that which enters into the function. In the same manner, taking the differential coefficient of $\frac{d.\mathrm{F}x}{dx}$, we shall find one term involving $(x-a)^m$, another $(x-a)^{m-1}$, and a third $(x-a)^{m-2}$; the last term will be $m(m-1)\mathrm{P}(x-a)^{m-2}$. Continuing the process, we shall see that each new differentiation produces again terms involving the same powers of $(x-a)$ that were contained in the function differentiated, with an additional term in which the power of $x-a$ is diminished by unity; thus, taking the successive differential coefficients, the term containing the lowest power of $x-a$ will be

for the first differentiation $m\mathrm{P}(x-a)^{m-1}$,
for the second . . . . $m(m-1)\mathrm{P}(x-a)^{m-2}$,
for the third . . . . $m(m-1)(m-2)\mathrm{P}(x-a)^{m-3}$,
for the nth . . . . $m(m-1)\ldots\ldots\mathrm{P}(x-a)^{m-n}$;

so that the differential coefficient of the rth order of $\mathrm{F}x$ will be of this form,

$$\frac{d^r.\mathrm{F}x}{dx^r}=\mathrm{X}(x-a)^m+\mathrm{X}'(x-a)^{m-1}+\mathrm{X}''(x-a)^{m-2}+\mathrm{X}'''(x-a)^{m-3}$$
$$+m(m-1)(m-2)\ldots\ldots\mathrm{P}(x-a)^{m-r}.$$

What we have said of Fx may be applied to ϕx, and we shall find, for the differential of the order r of the proposed function, a result of this form,

$$\frac{\frac{d^r.Fx}{dx^r}}{\frac{d^r.\phi x}{dx^r}}=\frac{X(x-a)^m+X(x-a)^{m-1}\ldots+m(m-1)\ldots P(x-a)^{m-r}}{Z(x-a)^n+Z'(x-a)^{n-1}\ldots.+n(n-1)\ldots Q(x-a)^{n-r}} \quad (44).$$

80. This being premised, we will consider three cases:

$1^o.\ m=n$; $2^o.\ m>n$; $3^o.\ m<n$.

If $m=n$, and the number of differentiations performed be also $=m$, the binomials $(x-a)^{m-r}$ and $(x-a)^{n-r}$ will be each reduced to $(x-a)^0$, *i. e.* to unity; whilst the other binomials $(x-a)^m$, $(x-a)^{m-1}$, &c.; $(x-a)^n$, $(x-a)^{n-1}$, &c. will become 0 on the supposition of $x=a$; thus all the terms, except the last of the numerator and the last of the denominator, will vanish, and the equation (44) will become

$$\frac{\frac{d^m.Fx}{dx^m}}{\frac{d^m.\phi x}{dx^m}}=\frac{m(m-1)(m-2)\ldots P}{m(m-1)(m-2)\ldots Q}=\frac{P}{Q}=\frac{Fx}{\phi x}.$$

In the second case, when we have $m>n$, if the number r of the differentiations performed be equal to n, the binomial $(x-a)^{n-r}$ is reduced to unity, its exponent $n-r$ being 0. The exponents $n-1$, $n-2$, &c.; $m-1$, $m-2$, &c. of the other binomials, being greater than $n-r$, are positive; and consequently the binomials are reduced each of them to 0, when x is made $=a$: on that hypothesis, therefore, all the terms vanish except that containing $(x-a)^{n-r}$, and the equation (44) is reduced to

$$\frac{\frac{d^n.Fx}{dx^n}}{\frac{d^n.\Phi x}{dx^n}}=\frac{0}{n(n-1)\ldots Q(x-a)^{n-r}}=\frac{0}{n(n-1)\ldots Q}=0.$$

This value, therefore, indicates that we have $m>n$, in which case the equation (43) is reduced to 0.

If, lastly, we have $m<n$, the number r of the differentiations performed being taken equal to m, all the terms will disappear except the one .... $m(m-1)\ldots P(x-a)^0$, and there will remain

$$\frac{\frac{d^m.Fx}{dx^m}}{\frac{d^m\phi x}{dx^m}}=\frac{m(m-1)\ldots P}{0}=\infty:$$

and this value, therefore, indicates that m is greater than n, in which case the second side of the equation (43) is infinite.

81. From what has preceded there results this rule: *When it is required to determine the true value of a fraction* $\frac{\mathrm{Fx}}{\phi\mathrm{x}}$, *which, on a particular value being given to the variable, becomes* $\frac{0}{0}$, *we must differentiate separately the two terms of the fraction, and examine then whether the results* $\frac{\mathrm{d}.\mathrm{Fx}}{\mathrm{dx}}$ *and* $\frac{\mathrm{d}.\phi\mathrm{x}}{\mathrm{dx}}$ *are also reduced to* 0, *for the proposed value of the variable; if this be the case, we must take the differential coefficients of the expressions* d.Fx *and* d.ϕx, *and see whether on the same hypothesis these are also reduced to* 0; *and continuing this process, if we find after a certain number of differentiations that the two terms of the fraction do not either of them vanish for the particular value of the variable, that last fraction will be the true value of* $\frac{\mathrm{Fx}}{\phi\mathrm{x}}$; *but if the numerator only become* 0 *for the value of* x, *the expression* $\frac{\mathrm{Fx}}{\phi\mathrm{x}}$ *is* 0; *and lastly, if it be only the denominator that vanishes for the value of* x, *the expression* $\frac{\mathrm{Fx}}{\phi\mathrm{x}}$ *will be infinite.*

82. Let us take, for example, the fraction

$$\frac{Fx}{\phi x}=\frac{x^3-b^3}{4(x-b)};$$

this fraction becoming $\frac{0}{0}$ when $x=b$, if we wish to have the true value, we must differentiate each of the two terms, when we shall obtain $\frac{3x^2}{4}$; and since the terms of this fraction do not either of them vanish on the supposition of $x=b$, the true value of the proposed fraction, when $x=b$, is $\frac{3b^2}{4}$.

83. We will take, for a second example, the fraction

$$\frac{x^3-3x+2}{x^4-6x^2+8x-3};$$

which is reduced to $\frac{0}{0}$ when $x=1$; and in order to obtain its true value, we must differentiate each of its terms, when we shall find

$$\frac{3x^2-3}{4x^3-12x+8};$$

in which fraction the two terms being again 0, on the hypothesis of $x=1$, we must differentiate again, and we shall obtain

$$\frac{6x}{12x^2-12}:$$

in this the denominator alone vanishes when we make $x=1$; and therefore the fraction proposed, on the hypothesis of $x=1$, is infinite.

84. If we apply the same rule to the fraction

$$\frac{a^x-b^x}{x},$$

which becomes $\frac{0}{0}$ on the hypothesis of $x=0$, we shall find, by differentiating the two terms of this fraction,

$$\frac{a^x\log a-b^x\log b}{1},$$

an expression, of which neither numerator nor denominator vanishes when $x=0$; and which consequently gives $\log a-\log b$ for the value of the proposed fraction, when $x=0$.

It is evident that the factor common to the two terms of the proposed fraction is $x-0$, or x: but how are we to recognise the factor x in a^x-b^x? To arrive at it we must observe that, according to art. (37),

$$a^x=1+A\frac{x}{1}+A^2\frac{x^2}{1.2}+\&c.$$

$$b^x=1+B^2\frac{x}{1}+B\frac{x^2}{1.2}+\&c.;$$

and, therefore, taking the difference.

$$a^x-b^x=(A-B)x+(A^2-B^2)\frac{x^2}{1.2}+\&c.,$$

whence we see that x is a factor of a^x-b^x.

85. It must not be supposed that the rule which we have just given will suffice for every case; the preceding demonstration is founded on the supposition of m and n being whole numbers; should they be fractional, we could never obtain, by successive differentiation, a term in which $x-a$ appears raised to the power 0, and consequently we could never, by the process hitherto employed, clear the fraction of the common factor.

For greater generality, then, let the expression be

$$\frac{Fx}{\phi x}=\frac{P(x-a)^{\alpha}+Q(x-a)^{\beta}+R(x-a)^{\gamma}+\&c.}{P'(x-a)^{\alpha'}+Q'(x-a)^{\beta'}+R'(x-a)^{\gamma'}+\&c.},$$

in which α, β, γ, &c. are positive and increasing, as also α', β', γ', &c. This expression becoming $\frac{0}{0}$ when $x=a$, we may, instead of changing x into a, change x into $a+h$, and make $h=0$, after having reduced; then the hypothesis will be the same as if we had immediately made $x=a$, and we shall have

$$\frac{\mathrm{F}x}{\varphi x}=\frac{\mathrm{P}h^{\alpha}+\mathrm{Q}h^{\beta}+\mathrm{R}h^{\gamma}+\&\mathrm{c.}}{\mathrm{P}'h^{\alpha'}+\mathrm{Q}'h^{\beta'}+\mathrm{R}'h^{\gamma'}+\&\mathrm{c.}}\;\ldots\ldots(45);$$

and α, α', being the least of the exponents in each of these series, we shall have three cases,

$$1^{\circ}.\ \alpha>\alpha';\quad 2^{\circ}.\ \alpha=\alpha';\quad 3^{\circ}.\ \alpha<\alpha'.$$

In the first case, dividing the two terms of the fraction by $h^{\alpha'}$, we have

$$\frac{\mathrm{F}x}{\varphi x}=\frac{\mathrm{P}h^{\alpha-\alpha'}+\mathrm{Q}h^{\beta-\alpha'}+\mathrm{R}h^{\gamma-\alpha'}+\&\mathrm{c.}}{\mathrm{P}'+\mathrm{Q}'h^{\beta'-\alpha'}+\mathrm{R}'h^{\gamma'-\alpha'}+\&\mathrm{c.}}\;\ldots\ldots\ldots(46):$$

and, by hypothesis, α is greater than α'; consequently the number $\alpha-\alpha'$ will be positive, and much more will $\beta-\alpha'$, $\gamma-\alpha'$, &c., be so also, since α, β, γ, &c., go on increasing. $\beta'-\alpha'$, $\gamma'-\alpha'$, &c., will likewise be positive; for α', β', γ', &c., going on increasing; α' is less than β', β' than γ', &c. This being premised, if we make $h=0$, all the terms on the second side of the equation (46) will vanish, except P′, and the equation will then be reduced to

$$\frac{\mathrm{F}x}{\varphi x}=\frac{0}{\mathrm{P}'}=0.$$

In the second case, in which $\alpha=\alpha'$, the term $\mathrm{P}h^{\alpha-\alpha'}$ is reduced to $\mathrm{P}h^{0}=\mathrm{P}$; and therefore, by inspecting the equation (46), we see that when $x=a$, $\frac{\mathrm{F}x}{\varphi x}$ is reduced to $\frac{\mathrm{P}}{\mathrm{P}'}$.

In the third and last case, in which α is less than α', dividing by h^{α}, we may write equation (45) thus:

$$\frac{\mathrm{F}x}{\varphi x}=\frac{\mathrm{P}+\mathrm{Q}h^{\beta-\alpha}+\mathrm{R}h^{\gamma-\alpha}+\&\mathrm{c.}}{\mathrm{P}'h^{\alpha'-\alpha}+\mathrm{Q}'h^{\beta'-\alpha}+\mathrm{R}'h^{\gamma'-\alpha}+\&\mathrm{c.}};$$

and we see that the hypothesis of $h=0$ reduces this equation to

$$\frac{\mathrm{F}x}{\varphi x}=\frac{\mathrm{P}}{0}=\infty.$$

86. Let us take, for example, the fraction

$$\frac{(x^2-3ax+2a^2)^{\frac{2}{3}}}{(x^3-a^3)^{\frac{1}{2}}},$$

which, when $x=a$, is reduced to $\frac{0}{0}$.

If in this fraction we put $a+h$ for x, it becomes

$$\frac{(h^2-ah)^{\frac{2}{3}}}{(3a^2h+3ah^2+h^3)^{\frac{1}{2}}}=\frac{(h-a)^{\frac{2}{3}}h^{\frac{2}{3}}}{(3a^2+3ah+h^2)^{\frac{1}{2}}h^{\frac{1}{2}}}$$

$$=\frac{(h-a)^{\frac{2}{3}}h^{\frac{2}{3}-\frac{1}{2}}}{(3a^2+3ah+h^2)^{\frac{1}{2}}}=\frac{(h-a)^{\frac{2}{3}}h^{\frac{1}{6}}}{(3a^2+3ah+h^2)^{\frac{1}{2}}};$$

and making $h=0$, we obtain

$$\frac{\mathrm{F}x}{\varphi x}=\frac{0}{(3a^2)^{\frac{1}{2}}}=0.$$

87. If a particular value of x render the two terms of the fraction $\frac{\mathrm{F}x}{\varphi x}$ infinite, we may divide each of the terms by $\mathrm{F}x\times\varphi x$, and we shall have

$$\frac{\mathrm{F}x}{\varphi x}=\frac{\frac{1}{\varphi x}}{\frac{1}{\mathrm{F}x}}=\frac{\frac{1}{\infty}}{\frac{1}{\infty}}=\frac{0}{0}.$$

88. If, lastly, we have a product mn, in which the hypothesis of $x=0$ renders one of the factors 0, and the other infinite; and m be the factor which becomes 0, n that which becomes infinite for the value of x; we may write the product thus

$$mn=\frac{m}{\frac{1}{n}},$$

and since $\frac{1}{n}$ will then be 0, the expression $\frac{m}{\frac{1}{n}}$ will be reduced to $\frac{0}{0}$.

On maxima and minima of functions of one variable.

89. We may, in the series of Taylor, give such a value to the increment h, that any one of the terms of the series shall

be greater than the sum of all those that follow. For, the series being represented by

$$y+\frac{dy}{dx}h+\frac{d^2y}{dx^2}\frac{h^2}{1.2}+\frac{a^3y}{dx^3}\frac{h^3}{1.2.3}+\&c.,$$

if we wish that $\frac{dy}{dx}h$, for instance, should become greater than the sum of all the other terms following, we may write the part of the series commencing from that term, thus:

$$\left(\frac{dy}{dx}+\frac{d^2y}{dx^2}\frac{h}{1.2}+\frac{d^3y}{dx^3}\frac{h^2}{1.2.3}+\&c.\right)h \ldots\ldots (47);$$

and since, when we make $h=0$, the part

$$\frac{d^2y}{dx^2}\frac{h}{2}+\frac{d^3y}{dx^3}\frac{h^2}{2.3}+\&c.$$

vanishes, it will be easily seen that by taking h exceedingly small, that part may be made as small as we please, and therefore be made less than $\frac{dy}{dx}$, which is independent of h. Let z be what $\frac{d^2y}{dx^2}\cdot\frac{h}{2}+\&c.$ becomes in this case: then the series will be reduced to $\left(\frac{dy}{dx}+z\right)h$; and since we have $\frac{dy}{dx}>z$, or, multiplying by h, $\frac{dy}{dx}h>zh$, it follows that the term $\frac{dy}{dx}h$ is greater than the sum of all the succeeding terms. The same may be proved for every term in respect of those that follow.

90. Let $y=\varphi x$ be an equation betwixt two variables. This may always be considered as the equation of a curve, in which the different values of the function y are the ordinates; and the function y is said to be at its minimum, when, after having been continually decreasing, it is on the point of commencing to increase.

Let, for example, MBN (fig. 9) be a curve whose equation is $y = b + cx^2$; we see, then, that the ordinates mp, $m'p'$, &c., go on continually diminishing up to the point B; but that after that point, the ordinates qn, $q'n'$ &c., go on continually increasing; so that the ordinate AB is the minimum of the function y.

91. Similarly the function y is said to be at its maximum when, after having been continually increasing, it is arrived at a point past which it begins to decrease.

The curve CDE, fig. 10, whose equation is $y = b - cx^2$, gives us an example of this case at the point D; for the ordinates immediately preceding and succeeding to AD are less than AD; and therefore the ordinate AD is a maximum.

92. There are some curves which have only a maximum, others which have only a minimum; there are some also which have both a maximum and a minimum, and others which do not allow of either.

We see, for example, that the curve, whose equation is $y = b + cx^2$, cannot have a maximum; for, from the nature of its equation, the ordinates go on continually increasing. The circle CBD, fig. 11, whose equation is

$$a^2 = (y - \beta)^2 + (x - \alpha)^2,$$

has both a maximum and a minimum, which correspond to the same abscissa AP; the maximum is PD, and the minimum PB.

93. When a function y of a variable x has a maximum or minimum, this maximum or minimum may be determined, if we know the abscissa corresponding to it. Suppose, for instance, that in a curve whose equation is $y = \varphi x$, we know the value a of the abscissa corresponding to the maximum or minimum; then we have only to make $x = a$ in the equation $y = \varphi x$, in order to determine the value of y, which is the maximum or minimum required.

94. Let, now, $y = fx$ be an ordinate PM, fig. 12, which is arrived at its maximum: if then the abscissa AP receive an

increment h, represented by PP′, and we draw also $PP''=h$, we shall have, for the conditions of PM being a maximum,

$$P'M' \angle PM, \qquad P''M'' \angle PM,$$

or,

$$f(x+h) \angle fx, \qquad f(x-h) \angle fx.$$

If, on the contrary, PM, fig. 13, be a minimum, representing the value of x, which corresponds to the minimum by AP, and taking $PP'=PP''=h$, we shall have for the conditions of the minimum,

$$P'M' \,7\, PM, \qquad P''M'' \,7\, PM,$$

or,

$$f(x+h) \,7\, fx, \qquad f(x-h) \,7\, fx.$$

Hence, when $f(x+h)$ and $f(x-h)$ are at the same time both less than fx, there will be a maximum; and if these functions be at the same time both greater than fx, there will be a minimum: if, lastly, one of these functions be greater, and the other less, than fx, there will be neither a maximum nor a minimum.

95. We must therefore investigate the cases in which these conditions can be fulfilled; and for this purpose we have, by Taylor's theorem,

$$f(x+h)=y+\frac{dy}{dx}h+\frac{d^2y}{dx^2}\cdot\frac{h^2}{1.2}+\frac{d^3y}{dx^3}\cdot\frac{h^3}{1.2.3}+\&c. \ldots\ldots (48);$$

in which series, if we change h into $-h$, we find also

$$f(x-h)=y-\frac{dy}{dx}h+\frac{d^2y}{dx^2}\frac{h^2}{1.2}-\frac{d^3y}{dx^3}\frac{h^3}{1.2.3}+\&c. \ldots\ldots (49).$$

In order, therefore, that $y=fx$ may be a maximum or a minimum, these two developments must be both less or both greater than y; but this cannot be, unless $\frac{dy}{dx}$ be $=0$. For by giving to h an exceedingly small value, we may always render

$\frac{dy}{dx}h$ greater than the sum of all the terms that follow; in which case the sign of $\frac{dy}{dx}h$ will be likewise the sign of $\frac{dy}{dx}h$, together with the following terms; so that, on this hypothesis, if $\frac{dy}{dx}h$ be positive in one of the developments (48) and (49), that development will be greater than y, and will be less than y, if $\frac{dy}{dx}h$ be negative. But the signs of $\frac{dy}{dx}h$ are different in these developments, and therefore, if $\frac{dy}{dx}h$ be positive in one, it must be negative in the other: whence it follows, that one of the quantities, $f(x+h)$ and $f(x-h)$, will be greater, and the other less than fx.

If, therefore, $\frac{dy}{dx}$ be not $=0$, there cannot be either a maximum or a minimum; but if $\frac{dy}{dx}=0$, then the developments (48) and (49) will be reduced to

$$f(x+h)=y+\frac{d^2y}{dx^2}\frac{h^2}{1.2}+\frac{d^3y}{dx^3}\frac{h^3}{2.3}+\&c.,$$

$$f(x-h)=y+\frac{d^2y}{dx^2}\frac{h^2}{1.2}-\frac{d^3y}{dx^3}\frac{h^3}{1.2.3}+\&c.;$$

in which case, the sign of the terms that follow y will depend on $\frac{d^2y}{dx^2}$, if only h be taken so small that that term may be greater than the sum of all those that follow; and since $\frac{d^2y}{dx^2}$ has the same sign in the two developments, it follows, that if $\frac{d^2y}{dx^2}$ be positive, the two functions of $x+h$ and $x-h$ will be both greater than fx; and in this case, therefore, fx will be a minimum. In the same manner it will be seen, that if $\frac{d^2y}{dx^2}$ be negative, fx will be a maximum.

96. To complete this theory, we must observe, that besides $\frac{dy}{dx}=0$, we may have also $\frac{d^2y}{dx^2}=0$; in which case we cannot have a maximum or minimum, unless also $\frac{d^3y}{dx^3}=0$. Then, taking h exceedingly small, the sign of the quantities following y will depend on $\frac{d^4y}{dx^4}$, and we may prove, as before, that if $\frac{d^4y}{dx^4}$ be positive, fx is a minimum; and if $\frac{d^4y}{dx^4}$ be negative, fx is a maximum; and so on. Generally, if the first coefficient, that does not vanish, be of an even order, there will be minimum when it is positive, and a maximum when it is negative.

97. For a first example, let us take the function $a-bx+x^2$; we shall have then

$$y=a-bx+x^2;$$

and differentiating and dividing by dx, we shall obtain

$$\frac{dy}{dx}=-b+2x,\ \frac{d^2y}{dx^2}=2;$$

where the positive value of $\frac{d^2y}{dx^2}$ shows us that the function has a minimum. To determine the abscissa corresponding to this minimum, we must equate to zero the value of $\frac{dy}{dx}$, which will give us $x=\frac{b}{2}$; and substituting this value of x in that of y, we shall find $y=a-\frac{b^2}{4}$ for the minimum sought.

98. Again, let the function be $a^4+b^3x-c^2x^2$; when dif-

ferentiating the equation $y=a^4+b^3x-c^2x^2$, and dividing by dx, we shall find,

$$\frac{dy}{dx}=b^3-2c^2x,\ \frac{d^2y}{dx^2}=-2c^2.$$

From the negative value of $\frac{d^2y}{dx^2}$, we see that there is a maximum in the function; the equation $b^3-2c^2x=0$, gives us $x=\frac{b^3}{2c^2}$ for the abscissa corresponding to that maximum; and substituting this value of x in that of y, we shall find for the maximum,

$$y=a^4+\frac{b^6}{4c^2}.$$

99. Let the equation be

$$y=3a^2x^3-b^4x+c^5;$$

proceeding as before, we shall find

$$\frac{dy}{dx}=9a^2x^2-b^4,\quad \frac{d^2y}{dx^2}=18a^2x;$$

equating to zero the value of $\frac{dy}{dx}$, we have

$$9a^2x^2-b^4=0,\ \text{whence}\ x=\pm\frac{b^2}{3a};$$

and substituting these two values of x successively in the value of $\frac{d^2y}{dx^2}$, we learn that the function has both a maximum and a minimum. The minimum corresponds to the abscissa $x=+\frac{b^2}{3a}$, the maximum to the abscissa $x=-\frac{b^2}{3a}$; and substituting these values in that of y, we shall find $y=c^5-\frac{2b^6}{9a}$, for the minimum, and $y=c^5+\frac{2b^6}{9a}$ for the maximum.

Application of the theory of maxima and minima to the solution of various problems.

PROBLEM I.

100. *To divide a number into two parts, such that the product of the parts shall be the greatest possible.*

Let a be the number, and x one of the parts; then $a-x$ will be the other, and $x(a-x)$ the quantity of which we have to determine the maximum.

Differentiating the equation $y=x(a-x)=ax-x^2$, and dividing by dx, we shall find

$$\frac{dy}{dx}=a-2x, \quad \frac{d^2y}{dx^2}=-2;$$

when the value of $\frac{dy}{dx^2}$ shows us, that the function really contains a maximum: had that coefficient appeared with a contrary sign, the problem would have been impossible. Equating therefore the value of $\frac{dy}{dx}$ to zero, we shall have $x=\frac{1}{2}a$, which informs us that the number a must be divided into two equal parts, in order that the product may be a maximum.

PROBLEM II.

101. *Of all the cylinders inscribed in a right cone, to determine that which has the greatest volume.*

Fig. 14. Let a, fig. 14, be the height SC of the cone, b the radius AC of the base, and x the distance SD from the vertex of the cone to the centre of the highest circle of the cylinder. Then the similar triangles SAC, SED will give us

$$\mathrm{SC} : \mathrm{AC} :: \mathrm{SD} : \mathrm{DE},$$

or,

$$a : b :: x : \mathrm{ED},$$

therefore

$$\mathrm{ED}=\frac{bx}{a}.$$

Let $1 : \pi$ be the ratio of the diameter to the circumference; then we know that the circle, whose radius is r, has for its surface πr^2; and therefore the circle EGF, which has $\frac{bx}{a}$ for its radius, has for its surface $\frac{\pi b^2}{a^2}x^2$. Multiplying this surface by the height DC of the cylinder, *i. e.* by $a-x$, we shall have $\frac{\pi b^2}{a^2}x^2(a-x)$ for the volume of the cylinder, and therefore the equation to be differentiated is

$$y=\frac{\pi b^2}{a^2}x^2(a-x)=\frac{\pi b^2}{a^2}(ax^2-x^3);$$

whence we deduce

$$\frac{dy}{dx}=\frac{\pi b^2}{a^2}(2ax-3x^2),\quad \frac{d^2y}{dx^2}=\frac{\pi b^2}{a^2}(2a-6x);$$

and equating to zero the value of $\frac{dy}{dx}$ we have

$$\frac{\pi b^2}{a^2}(2ax-3x^2)=0, \text{ or } 2ax-3x^2=0;$$

an equation which is the product of the factors x and $2a-3x$, and gives, consequently, $x=0$, or $x=\frac{2a}{3}$. The value $x=0$ cannot correspond to a maximum, since, on that hypothesis, $\frac{d^2y}{dx^2}$ is reduced to $\frac{2\pi b^2}{a^2}$, a positive number; and which therefore indicates a minimum; in fact, when $x=0$, the cylinder is reduced to the axis of the cone; for the higher the cylinder is, the more it is diminished in thickness.

The value $x=\frac{2a}{3}$ is consequently the only one that will answer the question; and on this hypothesis $\frac{d^2y}{dx^2}$ is reduced to $-\frac{2\pi b^2}{a}$, a negative number. If, therefore, we subtract . . . . $SD = x = \frac{2}{3}SC$ from the height of the cylinder, there will remain $CD = \frac{1}{3}SC$; whence it appears that *the cylinder of the greatest volume inscribed in the cone has for its height the third of that of the cone.*

PROBLEM III.

102. *To divide a straight line* AB (fig. 15) *into two parts,* AC *and* CB, *so that the product* $AC^3 \times CB$ *may be a maximum.*

Let us represent the straight line AB by a, and the part CB of that line by x; then the equation of the problem will be

$$y = x^3(a-x);$$

whence we deduce

$$\frac{dy}{dx} = 3ax^2 - 4x^3,\ \frac{d^2y}{dx^2} = 6ax - 12x^2;$$

and equating the value of $\frac{dy}{dx}$ to zero, we find $x=0$, or $x = \frac{3a}{4}$. This second value is the only one which can resolve the problem, since it reduces the value of $\frac{d^2y}{dx^2}$ to $-\frac{9a^2}{4}$, a negative result.

103. We may observe that when in the value of the differential coefficient $\frac{dy}{dx}$ we have a constant positive factor, this factor may be suppressed.

For if we have

$$\frac{dy}{dx} = A\varphi x,$$

we deduce from it

$$\frac{d^2y}{dx^2} = A\frac{d.\varphi x}{dx};$$

and this second equation serves only to make known to us the sign of the value of $\frac{d^2y}{dx^2}$; which sign, since A is a constant positive factor, will depend on that of $\frac{d.\varphi x}{dx}$; and therefore A may be suppressed in this equation. It may also be suppressed in the equation $\frac{dy}{dx} = A\varphi x$; for since we have to equate to zero the second side of this equation in order to determine x, the equation $A\varphi x = 0$ will give us $\varphi x = 0$; whence it follows that the constant A may be omitted altogether.

PROBLEM IV.

104. *A quantity of water of known bulk is to be put into a cylindrical vessel; required the dimensions of the vessel, so that its internal surface may be the least possible.*

Let V be the bulk of the water given, and x the radius of the base of the cylinder; then πx^2 will be the area of that base; and since the height of the cylinder, multiplied by its base, is equal to its bulk, we shall have

$$\textit{height of the cylinder} \times \pi x^2 = V,$$

whence we find

$$\textit{height of the cylinder} = \frac{V}{\pi x^2}.$$

Multiplying this height by the circumference of the base, which is $2\pi x$, we shall have

$$\frac{V}{\pi x^2}\times 2\pi x=\frac{2V}{x}$$

for the convex surface of the cylinder ; and adding to this surface πx^2, which is that of the base of the cylinder, the equation to be differentiated will be

$$y=\frac{2V}{x}+\pi x^2\ ;$$

from which we shall deduce

$$\frac{dy}{dx}=-\frac{2V}{x^2}+2\pi x,\ \frac{d^2y}{dx^2}=\frac{4V}{x^3}+2\pi\ ;$$

and the value of $\frac{dy}{dx}$, being equated to zero, gives

$$x=\sqrt[3]{\frac{V}{\pi}}.$$

This value, we see, answers to a mininum, since it renders the value of $\frac{d^2y}{dx^2}$ positive ; the radius of the base of the cylinder sought will therefore be $\sqrt[3]{\frac{V}{\pi}}$; and if we put this value in the expression for the height, we shall find for the height of the cylinder,

$$\frac{V}{\frac{\pi V^{\frac{2}{3}}}{\pi^{\frac{2}{3}}}}=\frac{V^{\frac{1}{3}}}{\pi^{\frac{1}{3}}}=\sqrt[3]{\frac{V}{\pi}}.$$

PROBLEM V.

105. *Of all the cones inscribed in a sphere, to determine that which has the greatest convex surface.*

Fig. 16. Suppose that the semi-circle AMB, fig. 16, makes a revolution around the axis AB ; then the chord AM will generate a cone, of which AP will be the height, and PM the radius of the base, and the expression for the convex surface of this

cone will be

$$\textit{circumference } \mathrm{PM} \times \tfrac{1}{2}\mathrm{AM} = 2\pi\mathrm{PM} \,.\, \tfrac{1}{2}\mathrm{AM} = \pi \,.\, \mathrm{PM} \,.\, \mathrm{AM}.$$

We have only, therefore, to determine PM and AM; for which purpose let $\mathrm{AB} = 2a$, $\mathrm{AP} = x$; then MP being a mean proportional between AP and PB, we have

$$x : \mathrm{PM} :: \mathrm{PM} : 2a - x;$$

and therefore

$$\mathrm{PM} = \sqrt{2ax - x^2};$$

AM also being a mean proportional between AP and AB, we have

$$x : \mathrm{AM} :: \mathrm{AM} : 2a$$

and therefore

$$\mathrm{AM} = \sqrt{2ax};$$

which values being substituted in the expression for the surface of the cone, we shall obtain

$$\begin{aligned} \textit{convex surface of cone} &= \pi\sqrt{2ax - x^2} \,.\, \sqrt{2ax} \\ &= \pi\sqrt{4a^2x^2 - 2ax^3}. \end{aligned}$$

The equation to be differentiated is therefore (art. 103)

$$y = \sqrt{4a^2x^2 - 2ax^3};$$

whence we deduce

$$\frac{dy}{dx} = \frac{4a^2x - 3ax^2}{\sqrt{4a^2x^2 - 2ax^3}},$$

or, suppressing the common factor x,

$$\frac{dy}{dx} = \frac{4a^2 - 3ax}{\sqrt{4a^2 - 2ax}} \quad \ldots\ldots \quad (50)$$

and equating this value of $\frac{dy}{dx}$ to zero, we shall have

$$4a^2 - 3ax = 0,$$

an equation satisfied by supposing

$$x = \frac{4a}{3}.$$

This value belongs to a minimum, as will be proved to us by the sign of $\frac{d^2y}{dx^2}$.

106. Before determining the value of this differential coefficient, we will explain a process, which in certain cases will abridge the calculations; and we will first observe, that when a function becomes 0 for some value given to x, it does not follow generally that the differential coefficient will be also 0; if, for example, we have the function x^2-5x+6, which becomes 0 when $x=2$, or $x=3$, the differential coefficient of this function, which is $2x-5$, does not become 0, on either of these hypotheses.

107. We may sometimes considerably shorten the operations which we have employed for discovering whether the function is susceptible of a maximum or a minimum. For suppose that we wished to determine the differential coefficient of the equation $\frac{dy}{dx}=\mathrm{XX}'$, in which X and X' are functions of x, and of which the first only becomes 0 for a particular value given to x; differentiating this equation, and dividing by dx, we shall get

$$\frac{d^2y}{dx^2}=\frac{\mathrm{X}d\mathrm{X}'}{dx}+\frac{\mathrm{X}'d\mathrm{X}}{dx},$$

and X, by hypothesis, being 0 for the value given to x, this equation is reduced to

$$\frac{d^2y}{dx^2}=\frac{\mathrm{X}'d\mathrm{X}}{dx},$$

which shows us that to obtain $\frac{d^2y}{dx^2}$, we have only to multiply the differential coefficient of the factor that vanishes by the other factor *.

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* This rule is not without exceptions, for $\frac{d\mathrm{X}}{dx}$ may be also 0. If, for in-

108. For example, if we wished to obtain the differential coefficient of the second order of $\frac{dy}{dx}=\frac{x-a}{\sqrt{x}}$ on the hypothesis of $x=a$; writing the equation thus,

$$\frac{dy}{dx}=\frac{1}{\sqrt{x}}(x-a),$$

we shall find that

$$\frac{d^2y}{dx^2}=\frac{1}{\sqrt{x}}\times\frac{d(x-a)}{dx}=\frac{1}{\sqrt{x}}.$$

109. We will return now to equation (50), from which we wished to determine the value of $\frac{d^2y}{dx^2}$, on the hypothesis of $x=\frac{4a}{3}$: and resolving the numerator into its factors, we shall have

$$\frac{dy}{dx}=\frac{ax(4a-3x)}{\sqrt{4a^2x^2-2ax^3}},$$

the second side of which may be written thus;

$$\frac{ax}{\sqrt{4a^2x^2-2ax^3}}\times(4a-3x),$$

and since, on the proposed hypothesis, the factor $4a-3x$ becomes 0, we shall have, art. 107,

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stance, we had $\frac{dy}{dx}=x^2(x-a)^2$, an equation which contains equal roots, the two terms of the value of $\frac{d^2y}{dx^2}$ will be each 0; and instead of suppressing the factor represented by $X.\frac{dX'}{dx}$, we must, art. 96, have recourse to the differential coefficients of the higher orders, to discover whether the function is susceptible of a maximum or minimum; if $\frac{dX'}{dx}$ be infinite, it will be the case of art. 87.

$$\frac{d^2y}{dx^2}=\frac{ax}{\sqrt{4a^2x^2-2ax^3}}\times\frac{d(4a-3x)}{dx}=\frac{-3ax}{\sqrt{4a^2x^2-2ax^3}},$$

whence, consequently, dividing the two terms of the fraction by x,

$$\frac{d^2y}{dx^2}=-\frac{3a}{\sqrt{4a^2-2ax}};$$

and putting in this expression the value of x, which is $\frac{4a}{3}$, we shall obtain

$$\frac{d^2y}{dx^2}=-\frac{3a}{\sqrt{4a^2-\frac{8a^2}{3}}}=-\frac{3a}{\sqrt{\frac{4a^2}{3}}};$$

which value being negative, that of x corresponds to a maximum.

PROBLEM VI.

110. *A point* C *(fig. 17) being given, in the angle* YAX, *to draw through that point a straight line* DE, *which shall meet the axes* AX, AY, *in such a manner that the length* DE, *of the straight line, shall be a maximum.*

Let $AI=a$, $IC=b$, $IE=x$; then the right-angled triangles ICE, ADE, give us

$$IE : IC :: AE : AD,$$

or

$$x : b :: a+x : AD;$$

therefore,

$$AD=\frac{b}{x}(a+x);$$

and consequently

$$AD^2=\frac{b^2}{x^2}(a+x)^2.$$

On the other hand,

$$AE^2=(a+x)^2;$$

which values being substituted in the formula

$$DE=\sqrt{\overline{AD}^2+AE^2},$$

we shall find

$$DE=\sqrt{\frac{b^2}{x^2}(a+x)^2+(a+x)^2}=\sqrt{\left(\frac{b^2}{x^2}+1\right)(a+x)^2};$$

and reducing the first factor under the root to the same denominator,

$$DE=\sqrt{\frac{b^2+x^2}{x^2}(a+x)^2}=\frac{a+x}{x}\sqrt{b^2+x^2}.$$

Considering this expression as the product of the factor $\frac{a+x}{x}$ by the factor $\sqrt{b^2+x^2}$, and differentiating by art. 14, we shall find

$$dy=\frac{a+x}{x}d.\sqrt{b^2+x^2}+\sqrt{b^2+x^2}.d\frac{a+x}{x};$$

performing the differentiations, we shall have

$$dy=\frac{a+x}{x}\frac{xdx}{\sqrt{b^2+x^2}}+\sqrt{b^2+x^2}\times-\frac{adx}{x^2};$$

reducing to the same denominator, by multiplying the two terms of the first fraction by x, and the two terms of the second by $\sqrt{b^2+x^2}$, we shall obtain

$$dy=\frac{a+x}{x^2}.\frac{x^2dx}{\sqrt{b^2+x^2}}+\frac{b^2+x^2}{x^2\sqrt{b^2+x^2}}\times-adx;$$

and collecting and reducing the terms of the numerator, and dividing by dx, there will result, lastly,

$$\frac{dy}{dx}=\frac{x^3-ab^2}{x^2\sqrt{b^2+x^2}},$$

when, the numerator being equated to zero, we shall find

$$x=\sqrt[3]{ab^2}.$$

To prove that this value answers to a minimum, we have only to put, art. 107, in place of the numerator, which is the factor that vanishes, its differential coefficient, and we shall have thus

$$\frac{d^2y}{dx^2}=\frac{3x^2}{x^2\sqrt{b^2+x^2}}=\frac{3}{\sqrt{b^2+x^2}},$$

a value essentially positive. We have not made the substitution of the value of x, since we see at once that the square x^2 is always positive.

PROBLEM VII.

111. *To find the greatest right-angled triangle that can be constructed on a given straight line.*

Fig. 18. Let a be the straight line, AB, fig. 18, and x one of the sides of the triangle; then the other will be $\sqrt{a^2-x^2}$, and the expression for the area of the triangle will be

$$\frac{x}{2}\sqrt{a^2-x^2}.$$

The equation of the problem will therefore be, art. 103,

$$y=x\sqrt{a^2-x^2}, \text{ or } y=\sqrt{a^2x^2-x^4},$$

whence we shall deduce

$$\frac{dy}{dx}=\frac{a^2x-2x^3}{\sqrt{a^2x^2-x^4}},$$

and this value, being equated to zero, gives

$$a^2x-2x^3=0, \text{ or } x(a^2-2x^2)=0,$$

an equation from which we derive

$$x=0, \text{ or } 2x^2=a^2.$$

But x cannot be 0, and we must therefore determine it from the second equation; which shows us that the two sides AC, BC, are equal.

By differentiating the factor a^2-2x^2, we find, art. 107,

$$\frac{d^2y}{dx^2}=\frac{x}{\sqrt{a^2x^2-x^4}}\cdot\frac{d(a^2-2x^2)}{dx}=-\frac{4x^2}{\sqrt{a^2x^2-x^4}};$$

and this result being negative, the hypothesis of $a^2-2x^2=0$ determines, for x, a value corresponding to a maximum.

On the geometrical signification of the differential coefficients.

112. We have seen already, art. 71, that $\frac{dy}{dx}$ represents the trigonometrical tangent of the angle, which a tangent drawn at the point, whose coordinates are x and y, makes with the axis of the abscissæ: but since this is the foundation of what is about to follow, we may demonstrate the proposition *à priori* in the manner following:

Let (fig. 4) $\text{PM}=y$, $\text{PP}'=h$; then drawing MQ parallel to the axis of the abscissæ, we have Fig. 4.

$$\text{M}'\text{P}'=f(x+h),$$

$$\text{M}'\text{Q}=f(x+h)-fx=\frac{dy}{dx}h+\frac{d^2y}{dx^2}\frac{h^2}{1.2}+\text{\&c.}$$

But

$$\text{MQ}:\text{M}'\text{Q}::1:\text{tang}\,\text{S}=\frac{\text{M}'\text{Q}}{\text{MQ}};$$

whence, putting for M′Q, MQ, their values, we shall have

$$\text{tang}\,\text{S}=\frac{\frac{dy}{dx}h+\frac{d^2y}{dx^2}\frac{h^2}{1.2}+\text{\&c.}}{h}=\frac{dy}{dx}+\frac{d^2y}{dx^2}\cdot\frac{h}{2}+\text{\&c.}:$$

and when we take the limit, h vanishes, and tangent S becomes tangent T; in that case therefore

$$\text{tang T}=\frac{dy}{dx}.$$

This being premised, if PM become a maximum, the tangent TM, being then parallel to the axis of the abscissæ, makes an angle 0 with that axis ; and since we have just seen that the trigonometrical tangent of the angle made by the tangent with the axis of x is represented by $\frac{dy}{dx}$, we must consequently, in this case, have $\frac{dy}{dx}=0$.

We might demonstrate in the same manner, that if PM were a minimum, in which case also the trigonometrical tangent would become 0, we ought to have $\frac{dy}{dx}=0$. Thus the condition expressed by the equation $\frac{dy}{dx}=0$ is that of the parallelism of the tangent at M to the axis of the abscissæ.

113. We will examine now under what circumstances $\frac{d^2y}{dx^2}$ is positive or negative ; and, with this view, we will consider, Fig. 20. first, the case in which the curve, fig. 20, turns its convexity towards the axis of the abscissæ.

Let $AP=x$, $PM=y$, $PP'=PP''=h$; and through the points M, M', draw the secant MM'S, and the straight lines MN, M'N', parallel to the axis of the abscissæ : then we shall have

$$M'O=M'P'-MP=f(x+h)-fx,$$

or

$$M'O=\frac{dy}{dx}h+\frac{d^2y}{dx^2}\frac{h^2}{1.2}+\&c.$$

But the similar triangles MM'O, MSN, give us

$$MO : MN :: M'O : SN,$$

or

$$h : 2h :: M'O : SN ;$$

therefore

$$SN = 2M'O ;$$

and substituting for M'O its value, we have

$$SN=2\frac{dy}{dx}h+2.\frac{d^2y}{dx^2}\frac{h^2}{1.2}+\&c.$$

On the other hand,

$$M''P''=f(x+2h),$$

from which, subtracting

$$NP''=PM,$$

we shall have

$$M''N=f(x+2h)-y=\frac{dy}{dx}2h+\frac{d^2y}{dx^2}\frac{4h^2}{1.2}+\&c.\,;$$

and taking from this value of M''N, that of SN, there will remain, (fig. 20.)

$$M''S=\frac{d^2y}{dx^2}h^2+\ \&c.\ \ldots\ (51).$$

In the case in which the curve, fig. 21, turns its concavity towards the axis of the abscissæ, to obtain M''S, we should have, on the contrary, to subtract the value of SN from that of M''N, which will give Fig. 21.

$$M''S=-\frac{d^2y}{dx^2}h^2+\ \&c.\ \ldots\ (52)\,;$$

and comparing these two values, (51), (52), of M''S, we see that in the one $\frac{d^2y}{dx^2}$ is preceded by the sign +, and in the other by the sign −.

This being premised, h may be so assumed that the sign of the first term in the development for M''S shall determine the sign of the whole of the development; and since the square h^2, which is essentially positive, cannot affect the sign of $\frac{d^2y}{dx^2}h^2$, the differential coefficient $\frac{d^2y}{dx^2}$ will itself determine the sign of the sum of all the terms in the value for M''S.

Considering, therefore, the equations (51), (52), relatively only to the signs which affect each side, we may suppress h^2, and the terms following $\frac{d^2y}{dx^2}$, when the equations will become

$$\mathrm{M''S} = +\frac{d^2y}{dx^2},\ \mathrm{M''S} = -\frac{d^2y}{dx^2},$$

whence we shall deduce

$$\left.\begin{aligned}\frac{d^2y}{dx^2} &= +\mathrm{M''S}\\ \frac{d^2y}{dx^2} &= -\mathrm{M''S}\end{aligned}\right\}\ldots\ldots (53).$$

Fig. 20. If now y be considered as a positive quantity, M''S (fig. 20), falling on the same side as y, will be positive; and the first of the equations (53) shows us therefore that, when the curve is convex to the axis of the abscissæ, $\frac{d^2y}{dx^2}$ is positive.

Considering next the second of the equations (53), and the
Fig. 21. fig. 21 which belongs to it, we shall see that $-$M''S represents a line which is of a sign contrary to that of y; and that consequently $\frac{d^2y}{dx^2}$ is negative in the case of the fig. 21, or when the curve is concave to the axis of the abscissæ.

114. The curve has hitherto been supposed to lie above the axis of the abscissæ; let us see now what takes place when it
Fig. 67. extends below that axis, as in the fig. 67. It is certain, then, from what has been already proved, that since the curve at M is convex to the axis of the abscissæ, $\frac{d^2y}{dx^2}$, and consequently MN is positive. But the straight lines MN and M'N', being situated on the same side of the tangent TT', ought to have the same sign; and since MN is positive, M'N' must be so also; whence it follows that at the point M', where the curve is concave to the axis of the abscissæ, $\frac{d^2y}{dx^2}$ will be of a sign contrary to that of the ordinate P'M', which is negative; the curve, on the contrary, would be convex, if y and $\frac{d^2y}{dx^2}$ were of the same sign. We may therefore say, generally,

that on whatever side the curve falls, $\frac{d^2y}{dx^2}$ has the same sign as y when the curve is convex to the axis of the abscissæ, and has a contrary sign when the curve is concave to the same axis.

The curve being convex or concave to the axis of the abscissæ accordingly as the ordinate is arrived at its minimum or its maximum, we see the reason why $\frac{d^2y}{dx^2}$ is positive in the first case and negative in the second.

115. There may also be a maximum or a minimum when $\frac{dy}{dx} = \infty$. To explain the nature of this condition, let $y = fx$ be the equation of a curve MN, fig. 22; then it is certain that if we give to x a value AP, that equation will determine the ordinate PM; and if, on the other hand, we resolve the equation in respect to y, and find $x = \varphi y$; when we put $y = \text{AP}'$ (the preceding value of y) the equation will give $x = \text{P}'\text{M}$. In this latter case y will be considered as the abscissa, and x the ordinate, and the same curve will be constructed, provided only that we draw the abscissæ y along the axis Ay, and consider the other axis as that of the ordinates. On this hypothesis, therefore, we may seek the maximum or minimum of x a function of y; and for this purpose we shall deduce from the proposed equation $\frac{dx}{dy} = \text{M}$, and put $\text{M} = 0$. But the equation $\frac{dx}{dy} = \text{M}$ gives us $\frac{dy}{dx} = \frac{1}{\text{M}}$, and we see therefore that when $\text{M} = 0$, $\frac{dy}{dx} = \infty$; thus the condition necessary, in order that we may have a maximum or minimum in this sense of the abscissæ, is that $\frac{dy}{dx} = \infty$. Fig. 22.

116. If, for example, we take the equation

$$y^2=ax-b,$$

we shall derive from it $\frac{dy}{dx}=\frac{a}{2y}$; which value being equated to zero will give $y=\infty$; and therefore the curve cannot have a maximum or minimum in respect of the ordinates, except at an infinite distance along the axis of x. Let us see now whether it has a limit in respect of the abscissæ (by limit denoting, generally, the maximum or minimum); and for this purpose we must suppose the value of $\frac{dy}{dx}$ infinite, which gives $\frac{a}{2y}=\infty$, a condition fulfilled by making $y=0$. On this hypothesis the value of $\frac{d^2x}{dy^2}$ is reduced to $\frac{2}{a}$, a positive result; and we see therefore that the value of $y=0$ corresponds to a minimum of x. We shall determine the value of this minimum by making $y=0$ in the proposed equation, which will reduce it to $ax-b=0$, whence we shall find $x=\frac{b}{a}$ for the minimum sought: this minimum is represented by AM in fig. 23.

117. Concluding this subject, we may observe that the equation $\frac{dy}{dx}=\infty$ indicates that the tangent MT, fig. 23, is that of a right angle, and that consequently MT is perpendicular to the axis of x.

General considerations on the singular points of curves.

118. The differential calculus may be of great service for finding the form of a curve of which the equation is given. The theory of maxima and minima has presented us already with the means of determining the limits in respect of the abscissæ and of the ordinates; but this will not be sufficient for making known to us the particular form of the curve. For

instance, the curves in figures 68, 69, and 70, have the same limits OC, OD, in respect of the ordinates, and OA, OB, in respect of the abscissæ, and yet are by no means similar to each other. What distinguishes the curve fig. 68 from the curve fig. 69 is that, in the latter, there is only a point of inflexion; this term being given to the point in which the curve from concave becomes convex, or from convex becomes concave. In the fig. 68 there are two points of inflexion, one at E, the other at G, and a point of reflexion at C, i. e. a point in which the curve at *once* stops its course. Fig. 68. Fig. 69.

119. In general, the points in which the curve undergoes any particular changes are termed *singular points;* and we see that if we have the means of determining where these points exist, it will be easy to follow the curve in its course. For example, if we know that the curve, fig. 70, has points of inflexion at E and H, and points of reflexion at F and G, we may form some idea of this curve by the following analysis:—In proceeding from the point A, which is a limit in respect of the abscissæ, the curve is at first concave to the axis of the abscissæ, and continues so up to E, where there is a point of inflexion, which from concave renders it convex. At the extremity of the convex part EF, the curve suspends its course at the point of reflexion F, beyond which it is still convex in the part FH, but becomes again concave beyond the point of inflexion H, and so reaches the point C, which is a limit in respect of the ordinates; lastly, from C to G and from A to G, the curve is composed of two arcs, CBG, ADG, which, being concave to the axis of the abscissæ, unite in a point of reflexion, and pass through the two limits B and D, the one in respect of the abscissæ, and the other in respect of the ordinates. Fig. 70.

120. From what has been said, we see how advantageous it would be to be able, by means of the equation of a curve, to determine the co-ordinates of its singular points. We have already explained the mode of finding the maxima and minima;

and it now remains for us to investigate the nature of the other singular points, which will be the object of the following sections.

On points of inflexion.

121. We have just seen that a point of inflexion is one in which the curve from convex becomes concave, or from concave becomes convex. The curve M′MM″, fig. 71, presents us a point of this description at M. Draw at this point a tangent TT′; then observing the different ordinates comprised betwixt M′P′ and MP, we shall see that the part M′N′ of the ordinate, lying betwixt the ordinate and the tangent, goes on continually diminishing, and at M will entirely vanish; whilst for the succeeding ordinates the part M″N″ of the ordinate will fall below the tangent, and will consequently change its sign; so that if M′N′ be positive, M″N″ will be negative. This condition we will proceed to express by an equation; and for this purpose let (fig. 71) $PP'=h=PP''$: then we have evidently Fig. 71.

$$M'N'=M'P'-N'P',$$

or

$$M'N'=f(x+h)-N'P' \quad . \; . \; . \quad (54).$$

To determine the analytical value of N′P′, we have

$$N'P'=MP+N'O,$$

or

$$N'P'=y+N'O \quad . \; . \; . \quad (55).$$

In regard to N′O, the right-angled triangle N′MO gives us

$$N'O=MO \,.\, \text{tang}\, N'MO\,;$$

but we have seen, art. 71, that the angle N′MO, formed by the tangent at M with a parallel to the axis of x, has $\frac{dy}{dx}$ for its trigonometrical tangent; replacing, therefore, tang N′MO by

$\frac{dy}{dx}$, and putting h in place of MO, we shall have

$$N'O = h \cdot \frac{dy}{dx}.$$

Substituting this value in equation (55), and putting then the value of N′P′ in equation (54), we shall obtain

$$M'N' = f(x+h) - y - \frac{dy}{dx}h \quad . \ . \ . \ (56).$$

Without having to calculate anew the value M″N″, we may deduce it from that of M′N′; for, if we suppose the ordinate to retire in a direction parallel to itself, M′N′ will become M″N″, when h is changed into $-h$; and giving, therefore, this value to h in the equation (56), we shall obtain

$$M''N'' = f(x-h) - y + \frac{dy}{dx}h \quad . \ . \ . \ (57).$$

Replacing now the expressions $f(x+h)$ and $f(x-h)$ by their developments, we shall have

$$M'N' = \left(y + \frac{dy}{dx}h + \frac{d^2y}{dx^2}\frac{h^2}{1.2} + \frac{d^3y}{dx^3}\cdot\frac{h^3}{1.2.3} + \&c.\right) - y - \frac{dy}{dx}h,$$

$$M''N'' = \left(y - \frac{dy}{dx}h + \frac{d^2y}{dx^2}\frac{h^2}{1.2} - \frac{d^3y}{dx^3}\cdot\frac{h^3}{1.2.3} + \&c.\right) - y + \frac{dy}{dx}h:$$

and by reducing, these equations will become

$$M'N' = \frac{d^2y}{dx^2}\frac{h^2}{1.2} + \frac{d^3y}{dx^3}\frac{h^3}{1.2.3} + \&c. \quad . \ . \ . \ . \ . \ (58),$$

$$M''N'' = \frac{d^2y}{dx^2}\frac{h^2}{1.2} - \frac{d^3y}{dx^3}\frac{h^3}{1.2.3} + \&c \quad . \ \ . \ (59).$$

In order now that there may be a point of inflexion at M, it is necessary that when we give to h an exceedingly small value, the lines M′N′ and M″N″ should fall one above and the other below the line TT′, and that consequently M′N′ and M″N″ should have different signs. But this is not possible, unless the first term in the series (58) and (59) be 0; for if that term be not 0, then we may give to h a value so small that the term $\frac{d^2y}{dx^2}\frac{h^2}{1.2}$ shall be greater than the sum of all the terms following, and therefore the sign of that term be the sign of the whole series; and since that term is the same in the two series, it follows that M′N′ and M″N″ would in this case have the same sign: in order, therefore, that M′N′ and M″N″ may have different signs, we must have

$$\frac{d^2y}{dx^2}\frac{h^2}{1.2}=0, \text{ or } \frac{d^2y}{dx^2}=0.$$

122. If it should happen that the same value of x, which makes $\frac{d^2y}{dx^2}$ vanish, make also $\frac{d^3y}{dx^3}$ vanish, then, in order that there may be a point of inflexion, $\frac{d^4y}{dx^4}$ must become 0 likewise; and if in this case $\frac{d^5y}{dx^5}$ result 0, $\frac{d^6y}{dx^6}$ must result 0 also: and generally the last differential coefficient that vanishes must be of an even order.

123. If the value of x, which is the same in the developments (58) and (59), be such that $\frac{d^2y}{dx^2}$ become infinite, the two developments will be so likewise; and we can then conclude nothing from the preceding demonstration, which rests on the supposition of these developments being possible. In this case we must observe that the condition $\frac{d^2y}{dx^2}=0$ indicates, generally, that $\frac{d^2y}{dx^2}$ ought to change its sign at the point of inflexion, which agrees with what was proved in art. **113**: but this change of sign may also take place in passing through infinity; for let

$$\frac{d^2y}{dx^2}=\frac{b^2}{x-a};$$

if, then, we substitute successively for x the values

$x=a-h$, we shall find $\frac{d^2y}{dx^2}=-\frac{b^2}{h}$,

$x=a$ $\qquad\qquad\qquad\qquad\quad$ $\frac{d^2y}{dx^2}=\ \ \infty$,

$x=a+h$, $\qquad\qquad\qquad\quad$ $\frac{d^2y}{dx^2}=+\frac{b^2}{h}$,

where we see that it is the denominator of the value of $\frac{d^2y}{dx^2}$, which produces the change of sign in the differential coefficient, after passing the point of inflexion.

124. Hence it follows, that if there be a point of inflexion in a curve, we must have, for the abscissa of that point,

$$\frac{d^2y}{dx^2}=0\ ;\ \text{or}\ \frac{d^2y}{dx^2}=\infty\ .$$

When, therefore, we have ascertained that one of these conditions is fulfilled, we must successively augment and diminish the abscissa of the point which fulfils the condition by a very small quantity h; and if, for these new values of x, $\frac{d^2y}{dx^2}$ has different signs, we may then conclude that there is a point of inflexion: for when $\frac{d^2y}{dx^2}$ is positive, the curve is convex to the axis of the abscissæ, and concave to that axis when $\frac{d^2y}{dx^2}$ is negative; but it is by this change from convex to concave, or from concave to convex, that the curve manifests its point of inflexion.

125. To give an application of this theory, let us examine whether there is a point of inflexion in the curve whose equation is

$$y=b+2(x-a)^3\ .\ .\ .\ .\ (60).$$

The differentiation gives us

$$\frac{dy}{dx}=3.2.(x-a)^2,\ \frac{d^2y}{dx^2}=12(x-a),\frac{d^3y}{dx^3}=12;$$

and in order that there may be a point of inflexion, there must be some value of x, which makes $\frac{d^2y}{dx^2}=0$. Now x being a variable quantity, we may determine one of its values by the condition that $12(x-a)=0$, when we shall obtain $x=a$ for the abscissa that may belong to a point of inflexion. To assure ourselves of the existence of this point, diminish the abscissa a by a very small quantity h, and substitute $a+h$ for x, when we shall find that for the point M′ (fig. 72), whose abscissa is $a-h$, we have $\frac{d^2y}{dx^2}=-12h$; substitute, then, $a+h$ for x, and we shall find that the point M″, whose abscissa is $a+h$, corresponds to $\frac{d^2y}{dx^2}=12h$. These two values of $\frac{d^2y}{dx^2}$, having different signs, show us that there is a point of inflexion at M.

The hypothesis of $x=a$ makes $\frac{dy}{dx}$ vanish, and consequently the tangent at the point of inflexion is parallel to the axis of x.

126. We may observe that we have not always the power of thus equating to zero the value of $\frac{d^2y}{dx^2}$; if, for instance, we wished to determine whether there were any points of inflexion in the curve which has for its equation

$$y=b+ax^2,$$

we should find, by differentiation,

$$\frac{dy}{dx}=2ax,\ \frac{d^2y}{dx^2}=2a;$$

and we see that this value of $\frac{d^2y}{dx^2}$, which contains no indeterminate quantity, cannot be equated to zero; and that consequently the curve cannot have a point of inflexion; a result to which we must attend, since the curve is a parabola. The value of $\frac{d^2y}{dx^2}$ shows us only that this parabola is constantly convex to the axis of the abscissæ.

127. For a third application, take the equation

$$y^3 = x^5,$$

which being resolved in respect of y, and then differentiated, we obtain

$$y = x^{\frac{5}{3}}, \quad \frac{dy}{dx} = \tfrac{5}{3}x^{\frac{2}{3}}, \quad \frac{d^2y}{dx^2} = \tfrac{5}{3} \cdot \tfrac{2}{3} \cdot \frac{1}{\sqrt[3]{x}}.$$

If, now, we sought to determine x from the equation . . . . . . $\frac{5}{3} \cdot \frac{2}{3} \frac{1}{\sqrt[3]{x}} = 0$, or $\frac{1}{\sqrt[3]{x}} = 0$, we could satisfy this equation only by making $x = \infty$; a value from which we could conclude nothing: but we are also at liberty to equate the value of $\frac{d^2y}{dx^2}$ to infinity, and since the equation $\frac{1}{\sqrt[3]{x}} = \infty$ is satisfied by making $x = 0$, this value of x shows us that there may be a point of inflexion at the origin. To convince ourselves that there is such a point, we might substitute successively for x the values $x = 0 + h$, and $x = 0 - h$, i. e. h and $-h$, and see if in these two cases $\frac{d^2y}{dx^2}$ produced results of different signs: but instead of performing these operations one after the other, we may accomplish them at once, by substituting for x the value $\pm h$, and then the differential coefficient of the second order will become

$$\frac{d^2y}{dx^2} = \pm \tfrac{5}{3} \cdot \tfrac{2}{3} \cdot \frac{1}{\sqrt[3]{h}}.$$

The higher value belongs to an abscissa greater than that at the point of inflexion, the lower to one less, and since these two values are of different signs, we may conclude from them that $x = 0$ corresponds to a point of inflexion A (fig. 73).

128. As a last example, we will take the curve which has for its equation

$$(y-b)^2 = x^3.$$

This equation gives us

$$y = b \pm x^{\frac{3}{2}}; \quad \frac{dy}{dx} = \pm \tfrac{3}{2} x^{\frac{1}{2}}, \quad \frac{d^2y}{dx^2} = \pm \tfrac{3}{2} \cdot \tfrac{1}{2} \cdot \frac{1}{\sqrt{x}};$$

and by making $x = 0$, we have $\frac{d^2y}{dx^2} = \infty$, which is a mark that we may meet with a point of inflexion at the origin. To ascertain whether this point really exists, we will make first $x = h$, and substitute this value in that of $\frac{d^2y}{dx^2}$, which becomes

$$\frac{d^2y}{dx^2} = \pm \tfrac{3}{2} \cdot \tfrac{1}{2} \frac{1}{\sqrt{h}}.$$

If then we make $x = -h$, the value of $\frac{d^2y}{dx^2}$ becomes imaginary, as does also that of y, which shows us that the curve has no existence for negative abscissæ; and thus, although $\frac{d^2y}{dx^2}$ be infinite at the origin, there is no point of inflexion. We shall shortly be able to recognize the origin A (fig. 74) as belonging to a class of points which have been comprised under the name of points of reflexion, or cusps; and which we shall proceed to examine more particularly in the following section. Fig. 74.

Points of reflexion.

129. When a curve stops in its course, and turns back again, we have a *point of reflexion;* and the reflexion is of the first species when the two branches have their convex sides opposed to each other, as in fig. 74; of the second species when the concave sides are both turned the same way, as in fig. 75. Fig. 74.

130. The curve stops thus, because beyond the point C of reflexion the values given to the abscissa determine imaginary ones for the ordinate, which supposes that $\frac{d^2y}{dx^2}$ contains some surd; and if, before the curve stops, $\frac{d^2y}{dx^2}$ give two values, one of the same sign as y, and the other of a contrary sign, this intimates that there are two branches of the curve which meet in the point c (fig. 74), the one convex to the axis of the abscissæ, the other concave, and by these characteristics, therefore, we should recognize a point of reflexion of the first species. If, on the contrary, the two values of $\frac{d^2y}{dx^2}$ have the same sign, the two branches which meet in C (fig. 75) must be concentric; and, consequently, the reflexions will, in this case, be of the second species.

131. For a first example, let us examine whether there are any points of reflexion in the curve which has for its equation

$$(y-x)^2=x^9.$$

This equation gives

$$y=x\pm x^4\sqrt{x}\,\ldots\ldots(61);$$

and we see that when we take x negative, y becomes imaginary, so that the curve stops at the origin, where $x=0$ and $y=0$: but yet this does not prove that there is at the origin a point of reflexion; for there might at that point be merely an arc of the curve, having its concavity always turned the same way, as is the case at the vertex of the hyperbola: thus, to determine whether the value of $x=0$ corresponds to a point of reflexion, we must know what the differential coefficient of the second order becomes near the origin.

Now, by differentiating the equation

$$y=x\pm x^{\frac{9}{2}},$$

and dividing by dx, we find

$$\frac{dy}{dx}=1\pm\tfrac{9}{2}x^{\frac{7}{2}}\ \ldots\ldots\quad(62),$$

$$\frac{d^2y}{dx^2}=\pm\tfrac{9}{2}\cdot\tfrac{7}{2}x^{\frac{5}{2}}=\pm\tfrac{9}{2}\cdot\tfrac{7}{2}x^2\sqrt{x};$$

and to determine whether the curve is concave or convex near the point where it stops its course, we must increase the abscissa of that point by a very small quantity h, by making $x=0+h=h$, and substitute this value of x in that of $\frac{d^2y}{dx^2}$, when we shall find

$$\frac{d^2y}{dx^2}=\pm\tfrac{9}{2}\cdot\tfrac{7}{2}h^2\sqrt{h}.$$

These two values with different signs indicate therefore that there are two branches; the one, AM (fig. 76), which is convex to the axis of the abscissæ, the other, AN, which is concave to the same axis: and consequently the origin is a point of reflexion of the first species.

132. For a second example, take the equation

$$(y-b)^2=(x-a)^3.$$

This equation gives us

$$y=b\pm\sqrt{(x-a)^3}\quad\ldots\ldots(63);$$

and if we make $x=a$, we find $y=b$; but if we give to x values less than a, those of y become imaginary; for on putting $a-h$ for x, we find

$$y = b \pm \sqrt{-h^3} = b \pm h\sqrt{-h},$$

an imaginary value; and the curve, therefore, stops at the point C (fig. 71), whose coordinates are a and b.

To know in what manner the branches proceed beyond the point C, we must substitute for x the value $a+h$ in that of $\frac{d^2y}{dx^2}$, when we shall obtain

$$\frac{d^2y}{dx^2} = \pm\frac{3}{4\sqrt{h}}.$$

The higher sign of $\frac{d^2y}{dx^2}$ points out a branch CM, which is convex to the axis of x; the lower sign points out a branch CN, which is concave to the same axis; and there is therefore at C a point of reflexion of the first species.

133. As a third example, take the curve whose equation is

$$y = ax^2 \pm bx^2\sqrt{x}.$$

If now we make $x=0$, we find $y=0$; but for a negative value of x, y becomes imaginary; and the curve therefore stops at the origin. Let us examine what $\frac{d^2y}{dx^2}$ becomes in this case; for which purpose, by writing the equation of the curve in the manner following,

$$y = ax^2 \pm bx^{\frac{5}{2}}$$

we shall obtain

$$\frac{dy}{dx} = 2ax \pm \tfrac{5}{2}bx^{\frac{3}{2}},$$

$$\frac{dy^2}{dx^2} = 2a \pm \tfrac{5}{2}\cdot\tfrac{3}{2}b\sqrt{x};$$

when giving to x an exceedingly small value, represented by h, the part $\frac{5}{2}\cdot\frac{3}{2}b\sqrt{x}$ of the value of $\frac{d^2y}{dx^2}$ will be less than the part

$2a$, and consequently the two values of $\frac{d^2y}{dx^2}$ given by the equation

$$\frac{d^2y}{dx^2}=2a\pm\tfrac{5}{2}\cdot\tfrac{3}{2}b\sqrt{h},$$

will be positive.

It follows, therefore, that at the origin there are two branches, both concave to the axis of x; and we have, consequently, at the origin, a point of reflexion of the second species.

134. The points of reflexion belong to a class of points comprised under the denomination of multiple points.

Multiple points.

135. Those points are called *multiple points* in which several branches of a curve meet. A multiple point is double when it is at the intersection of two branches, triple when at the intersection of three, and so on.

Fig. 77. 136. Let A (fig. 77) be a double point, formed by the intersection of the two branches of the curve AB, AC, to which AT and AT' are drawn tangents. If, now, the equation of the curve, freed from surds, be represented by $F(x, y)=0$, the differential of this equation, put under the form $Pdx+Qdy=0$, will contain no surd quantity, since no such quantities can be introduced by the differentiation of a rational function; it follows, therefore, that P and Q will be rational quantities. This being premised, the above equation gives us

$$\frac{dy}{dx}=-\frac{P}{Q}\quad\ldots\quad(64)$$

and since at the point in question there are two tangents $\frac{dy}{dx}$, and consequently $\frac{P}{Q}$ must necessarily have two different values; a condition which would be fulfilled if $\frac{P}{Q}$ involved a surd quantity; but this is impossible, since we have already seen that $\frac{P}{Q}$ is rational: in this case, therefore, the principles of algebra must conduct us to a result which avoids this contradiction; and this will be when $\frac{P}{Q}$ appears under the form $\frac{0}{0}$; for we know that $\frac{0}{0}$ is the symbol of an indeterminate quantity, and consequently susceptible of several values.

137. To show how this theorem may be demonstrated, suppose, for an instant, that α and α' represent the two values of the trigonometrical tangent of the curve at the multiple point; these two values, then, must satisfy the equation

$$P+Q\frac{dy}{dx}=0.$$

and will give

$$P+Q\alpha=0,\ P+Q\alpha'=0;$$

whence, subtracting the last equations one from the other, we obtain

$$Q(\alpha-\alpha')=0.$$

Now the factor $\alpha-\alpha'$, being composed of two unequal quantities, cannot be 0; it follows, therefore, that $Q=0$, which reduces the equation $P+Q\alpha=0$ to $P=0$; and by means of these values of P and Q, the equation $P+Q\frac{dy}{dx}=0$, or . . . $\frac{dy}{dx}=-\frac{P}{Q}$, becomes

$$\frac{dy}{dx}=\frac{0}{0}.$$

138. If instead of two branches meeting in a point, we had a greater number, it would be sufficient to consider only two, to show that at the point of intersection of all the branches

$\frac{dy}{dx}$ must be $=\frac{0}{0}$; we cannot arrive so easily at the same conclusion, when several branches of the curve have a common tangent; but in this case, also, it may be proved that $\frac{dy}{dx}$ must appear under the form $\frac{0}{0}$. As however the demonstration of this theorem is founded on the consideration of the contact of curves, we will reserve it for art. 170, when we shall have discussed the subject of osculating curves.

139. It must be observed, that the demonstration of art. 137 being founded on the supposition that the primitive equation has been cleared of surd quantities, if we differentiate that equation without having so cleared it, it may happen that an equation which allows of multiple points will not give $\frac{dy}{dx}=\frac{0}{0}$. The equation of art. 131, for instance, comes under this case; it has a double point at the origin, and yet if we make $x=0$, the equation (62) is reduced to $\frac{dy}{dx}=1$.

140. We will add, lastly, that though the equation $\frac{dy}{dx}=\frac{0}{0}$ holds good for a multiple point, it does not follow that it subsists only for a point of that description; for the preceding demonstration does not at all infer that the property is confined solely to such points. Thus all that we can conclude from this is, that the reduction of $\frac{dy}{dx}$ to $\frac{0}{0}$ indicates that there may be a multiple point.

141. What has been said will be sufficient to point out to us the means of determining whether there exist any multiple points in a curve whose equation is given. Let u, for instance, be the equation; we must deduce from it, by differentiating, $\mathrm{P}dx+\mathrm{Q}dy=0$, and see whether the same values of x and y

will satisfy the proposed equation, and also the equations $P=0$, $Q=0$; if this be the case, it shows that the values of x and y may belong to a multiple point, and then, by examining the curve in the neighbourhood of that point, we shall discover whether it is really a multiple point or not.

Conjugate points.

142. Suppose we have a curve such, that whilst for a particular point there are two real coordinates, the coordinates for all adjoining points are imaginary; those coordinates then will determine a point entirely detached from the curve, and to which has been given the name of *isolated point*, or *conjugate point*.

Let now $y=fx$ represent the equation of a curve which has a conjugate point. If a and b be the coordinates of that point, the coordinates for at least the adjoining points must be imaginary, or it could not be isolated; and, consequently, if we suppose that the abscissa a is increased by a small quantity h, the corresponding ordinate, represented by $f(a+h)$, must become imaginary.

Now the series of Taylor gives us, generally,

$$f(x+h)=y+\frac{dy}{dx}h+\frac{d^2y}{dx^2}\cdot\frac{h^2}{1.2}+\frac{d^3y}{dx^3}\,\frac{h^3}{1.2.3}+\&c.;$$

and if we make $x=a$, the corresponding ordinate must be b: whence, changing y into b, and representing by $\left(\frac{dy}{dx}\right)$, $\left(\frac{d^2y}{dx^2}\right)$, $\left(\frac{d^3y}{dx^3}\right)$, &c. the values of the differential coefficients on that hypothesis, we shall have

$$f(a+h)=b+\left(\frac{dy}{dx}\right)h+\left(\frac{d^2y}{dx^2}\right)\frac{h^2}{1.2}+\left(\frac{d^3y}{dx^3}\right)\frac{h^3}{1.2.3}+\&c.$$

In order therefore that $f(a+h)$ may be an imaginary quantity, one at least of the expressions $\left(\frac{dy}{dx}\right)$, $\left(\frac{d^2y}{dx^2}\right)$, &c., must be imaginary; i.e. the hypothesis of $x=a+h$ must render one of the differential coefficients imaginary; and if this condition be fulfilled, the curve may have a conjugate point.

For example, if we have the equation

$$y=\pm(x+b)\sqrt{x},$$

we shall find, by differentiating,

$$\frac{dy}{dx}=\pm\left(\frac{3}{2}\sqrt{x}+\frac{b}{2\sqrt{x}}\right),$$

which becoming imaginary when $x=-b$, and consequently $y=0$, we may presume that the point A (fig. 78), whose co-ordinates are $x=-b$, and $y=0$, is a conjugate point: we must determine whether it is really so or not, by successively increasing and diminishing the abscissa $-b$ by a quantity less than b; when we shall find that in each case y becomes imaginary; which shows therefore that the point in question is a conjugate point.

143. Conjugate points, like multiple points, manifest their existence by rendering the differential coefficient $\frac{dy}{dx}=0$. For the equation

$$Q\frac{dy}{dx}+P=0,$$

being differentiated and divided by dx, gives

$$Q\frac{d^2y}{dx^2}+\frac{dy}{dx}\cdot\frac{dQ}{dx}+\frac{dP}{dx}=0;$$

where we see that the term affected by $\frac{d^2y}{dx^2}$ has Q for its co-efficient; and if we differentiate again, we shall find that Q is still the coefficient of $\frac{d^3y}{dx^3}$, and so on; so that when we have arrived at the differential coefficient of the nth order, we shall have a result of the form

$$Q\frac{d^ny}{dx^n}+K=0\ .\ .\ .\ .\ .\ (65).$$

Now there must be at least one of the differential coefficients which becomes imaginary for some value of x, and which consequently must contain a surd quantity: representing, therefore, this coefficient by $\frac{d^ny}{dx^n}$, the function of x which expresses this coefficient, must have more than one value. This is sufficient for us to conclude, as in art. 137, that $Q=0$, which reduces the equation $P+Q\frac{dy}{dx}=0$, to $P=0$; and it follows, therefore, that we must have $\frac{dy}{dx}=\frac{0}{0}$.

Osculating curves.

144. Let $y=\varphi x$ and $y=Fx$ be the equations to two curves which meet (fig. 24) in the point M, whose coordinates are $AP=x'$, $PM=y'$; we shall have then, for that point,

$$\varphi x'=Fx';$$

and supposing that x' becomes $x'+h$, the preceding equations will give

$$M'P'=\varphi(x'+h)=\varphi x'=\frac{d.\varphi x'}{dx'}h+\frac{d^2\varphi x'}{dx'^2}\frac{h^2}{1.2}+\&c.\ .\ .\ .\ .\ .\ (66),$$

$$M''P'=F(x'+h)=Fx'+\frac{d.Fx'}{dx'}h+\frac{d^2.Fx'}{dx'^2}\frac{h^2}{1.2}+\&c.\ .\ .\ .\ .\ (67).$$

If, now, all the corresponding terms of these developments be identically the same, M'P', M''P' will have the same values, and the two curves will coincide; but if we have only $\varphi x'=Fx'$, the curves, as we have just seen, will have merely a common point M; if, besides $\varphi x'=Fx'$, we have

$$\frac{d.\varphi x'}{dx'}=\frac{d.Fx'}{dx'},$$

the curves will then approach more nearly to each other ; and still more, if, in addition to these equations, we have also

$$\frac{d^2\varphi x'}{dx'^2} = \frac{d^2 \mathrm{F}x'}{dx'^2} ;$$

and so on in order ; for it is evident that the difference betwixt M″P′ and M′P′ will be the less, the greater be the number of terms respectively equal in their developments.

This being premised, let a, b, c, &c., be the constants in the equation $y = \mathrm{F}x$; we may then, without changing the nature of the curve, give arbitrary values to these constants. If, for instance, we have the equation $y^2 = mx + nx^2$, which is that of an ellipse ; this equation will always preserve the same form, and will therefore always belong to an ellipse, whatever be the values we give to m and n, (m and n being understood to vary only in magnitude, and not in sign, and never to become 0).

We may now, therefore, consider the constants a, b, c, &c., which enter into the equations

$$y' = \mathrm{F}x', \ \frac{d\mathrm{F}x'}{dx'} = \frac{d\varphi x'}{dx'}, \quad \frac{d^2\mathrm{F}x'}{dx''} = \frac{d^2\phi x'}{dx'^2}, \text{ \&c.,}$$

as arbitrary, and taking as many of these equations as there are constants, determine the constants by the condition that those equations are satisfied.

For example, if the equation $y = \mathrm{F}x$ contain three constants, a, b, c, we may put

$$\mathrm{F}x' = \varphi x', \quad \frac{d\mathrm{F}x'}{dx'} = \frac{d\phi x'}{dx'}, \quad \frac{d^2\mathrm{F}x'}{dx'^2} = \frac{d^2\varphi x'}{dx'^2},$$

deduce from these equations the values of a, b, c, in functions of x', y', $\frac{dy'}{dx'}$, $\frac{d^2y'}{dx'^2}$, and substitute them in the equation $y = \mathrm{F}x$; which will then possess this property, that when we put $x' + h$ for x, the equation (67), obtained by means of Taylor's formula, will have the three first terms of its second side respectively equal the three first terms of the second side of equation (66).

What we have said of an equation containing only three constants, will apply to one containing a greater number.

145. Let us take, for example, the case in which the equation $y=Fx$ represents that of a straight line; the equation $y=Fx$ will be then replaced by

$$y=ax+b \quad \ldots\ldots \quad (68);$$

and the equations of condition, necessary for the elimination of the constants a and b, will be

$$\varphi x' = ax'+b, \quad \frac{d\phi x'}{dx'}=a \quad \ldots\ldots \quad (69).$$

But since $\varphi x'$ represents the ordinate at M of a curve whose equation is $y=\phi x$, and x' corresponds to y', we may replace $\phi x'$ by y', and the equations (69) will become

$$y'=ax'+b, \quad \frac{dy'}{dx'}=a;$$

whence, eliminating a, we obtain

$$y'=\frac{dy'}{dx'}\cdot x'+b,$$

and substituting the value of b given by this equation, and that of a in the equation (68) to the straight line, it becomes

$$y-y'=\frac{dy'}{dx'}(x-x') \quad \ldots\ldots \quad (70).$$

In this equation we shall recognize that of a tangent MT (fig. 5) at the point M, whose coordinates are x' and y': why the line MT should be such a tangent will be seen shortly.

146. Returning now to the preceding theory; and agreeing, for the sake of brevity, to denominate curves by their equations, we have seen (art. 144) that if the curves $y=\varphi x$ and $y=Fx$ have only a common point, and x', y', be the coordinates of that point, we shall have the equation of condition

$Fx' = \phi x'$; but that if we determine two constants of the equation $y = Fx$, by the conditions $Fx' = \phi x'$, and $\frac{dFx'}{dx'} = \frac{d\varphi x'}{dx'}$, the curves will begin to approach each other.

Let $y = fx$ represent what $y = Fx$ becomes after we have substituted the values of these two constants; then $y = fx$ will be an osculate of the first order to the curve $y = \varphi x$; and if (always by virtue of the arbitrary values that may be given to the constants) we have eliminated three of the constants of the equation $y = Fx$, by means of the following equations:

$$Fx' = \phi x', \ \frac{dFx'}{dx'} = \frac{d\varphi x'}{dx'}, \ \frac{d^2Fx'}{dx'^2} = \frac{d^2\phi x'}{dx'^2} \ \ldots\ldots\ldots \ (71);$$

and $y = \psi x$ represent what $y = Fx$ becomes after this substitution, the curve $y = \psi x$ will be an osculate of the second order to the curve $y = \phi x$, which it will approach still nearer than $y = fx$ does, and so on, in order; so that for an osculate of the nth order, we shall have the equations

$$Fx' = \varphi x', \ \frac{dFx'}{dx'} = \frac{d\phi x'}{dx'}, \ \frac{d^2Fx'}{dx'^2} = \frac{d^2\phi x'}{dx'^2} \ldots \frac{d^nFx'}{dx'^n} = \frac{d^n\phi x'}{dx'^n} \quad (72).$$

147. We will proceed to show, that of two osculates which we have thus obtained, by giving arbitrary values to the constants of the same equation, the osculate of an inferior order cannot pass between the other and the curve, in respect to which the osculation takes place. For example, let MB (fig. 24) be the curve $y = \varphi x$, and MC its osculate $y = \psi x$ of the second order; we have then to demonstrate that the osculate $y = fx$ of the first order cannot pass betwixt the curves MB and MC.

For this purpose, by putting $x' + h$ in place of x, in these equations, we shall find

$$P'M' \text{ or } \varphi(x' + h) = \varphi x' + \frac{d\varphi x'}{dx'}h + \frac{d^2\varphi x'}{dx'^2}\frac{h^2}{1.2} + \frac{d^3\phi x'}{dx'^3}\frac{h^3}{2.3} + \&c.,$$

$$P'M'' \text{ or } \psi(x'+h)=\psi x'+\frac{d\psi x'}{dx'}h+\frac{d^2\psi x'}{dx'^2}\,\frac{h^2}{1.2}+\frac{d^3\psi x'}{dx'^3}\,\frac{h^3}{2.3}+\&c.,$$

$$f(x'+h)=fx'+\frac{dfx'}{dx'}h+\frac{d^2fx'}{dx'^2}\cdot\frac{h^2}{1.2}+\frac{d^3fx'}{dx'^3}\cdot\frac{h^3}{2.3}+\&c.,$$

and $y=\psi x$ being an osculate of the second order to $y=\varphi x$, we must have

$$\psi x'=\phi x',\ \frac{d\psi x'}{dx'}=\frac{d\phi x'}{dx'},\ \frac{d^2\psi x'}{dx'^2}=\frac{d^2\varphi x'}{dx'^2};$$

$y=fx$ being an osculate of the first order to $y=\varphi x$, we must have also

$$fx'=\phi x',\ \frac{dfx'}{dx'}=\frac{d\varphi x'}{dx'};$$

from which equations we have therefore

$$\phi x'=\psi x'=fx',$$

$$\frac{d\varphi x'}{dx'}=\frac{d\psi x'}{dx'}=\frac{dfx'}{dx'},$$

and only

$$\frac{d^2\varphi x'}{dx'^2}=\frac{d^2\psi x'}{dx'^2}.$$

Make

$$\phi x'+\frac{d\varphi x'}{dx'}h=\mathrm{K},$$

$$\tfrac{1}{2}\cdot\frac{d^2\varphi x'}{dx'^2}=\mathrm{V};$$

then the three preceding developments may be written thus,

$$P'M' \text{ or } \varphi(x'+h)=\mathrm{K}+\mathrm{V}h^2+\frac{d^3\varphi x'}{dx'^3}\,\frac{h^3}{2.3}+\&c.,$$

$$P'M'' \text{ or } \psi(x'+h)=\mathrm{K}+\mathrm{V}h^2+\frac{d^3\psi x'}{dx'^3}\cdot\frac{h^3}{2.3}+\&c.,$$

$$f(x'+h)=\mathrm{K}+\frac{d^2fx'}{dx'^2}\,\frac{h^2}{1.2}+\frac{d^3fx'}{dx'^3}\cdot\frac{h^3}{1.2.3}+\&c.;$$

and observing that all the terms, commencing from that which is affected by h^3, have h^3 for a common factor, we may suppose

$$\frac{d^3\phi x'}{dx'^3}\frac{h^3}{2.3}+\&c. = Mh^3;$$

whence, making similar reductions in the other equations, we shall have

$$\begin{aligned}\phi(x'+h) &= K+Vh^2+Mh^3,\\ \psi(x'+h) &= K+Vh^2+Nh^3,\\ f(x'+h) &= K+\tfrac{1}{2}\frac{d^2fx'}{dx'^2}h^2+Ph^3.\end{aligned}$$

Now the curves $y=fx$ and $y=\psi x$ being osculating curves, one of the first and the other of the second order, V must necessarily differ from $\frac{1}{2}\frac{d^2fx'}{dx'^2}$; and we can therefore make only two hypotheses respecting V, viz.

$$V \angle \tfrac{1}{2}\frac{d^2fx'}{dx'^2}, \text{ or } V \urcorner \tfrac{1}{2}\frac{d^2fx'}{dx'^2}.$$

If V be less than $\frac{1}{2}\frac{d^2fx'}{dx'^2}$, let Z be the excess of $\frac{1}{2}\frac{d^2fx'}{dx'^2}$ over V, then we shall have

$$V+Z=\tfrac{1}{2}\frac{d^2fx'}{dx'^2},$$

where Z is a positive quantity; but if, on the contrary, V be greater than $\frac{1}{2}\frac{d^2fx'}{dx'^2}$, Z will be negative.

Substituting this value of $\frac{1}{2}\frac{d^2fx'}{dx'^2}$ in that of $f(x'+h)$, and observing that h^2 is a common factor, our three developments will now become

$$\begin{aligned}\phi(x'+h) &= K+(V+Mh)h^2,\\ \psi(x'+h) &= K+(V+Nh)h^2,\\ f(x'+h) &= K+(V+Z+Ph)h^2;\end{aligned}$$

and by making h exceedingly small, the quantity Z, which is independent of h, may become greater than the expressions Mh and Nh, which tend to 0.

In this case, if Z be positive, $f(x'+h)$ is greater than $\varphi(x'+h)$ and $\psi(x'+h)$, and we have therefore $f(x'+h)$ or $P'M'''$ (fig. 24) greater than either $P'M'$ or $P'M''$, which shows that the curve $y=fx$, represented by MM''', cannot pass between the other two.

If, on the contrary, Z be negative, we have $f(x'+h)$, or $P'M^{\text{IV}}$ less than $P'M'$ and $P'M''$; and the curve MM^{IV} being then that which approaches nearest to the axis of x, cannot lie betwixt the two others.

148. We can now explain why the straight line (fig. 5) which, art. 145, is an osculate of the first order, is a tangent to the curve; for it follows from our theory, that betwixt that straight line and the curve, we cannot draw any other straight line, which is a property of the tangent.

The tangent is said to have a *contact* of the first order with the curve; and generally an osculate of the order n has a contact of the same order with the curve to which it is an osculate; thus when we have, betwixt the two curves, the equations

$$\varphi x' = Fx', \quad \frac{d\varphi x'}{dx'} = \frac{dFx'}{dx'}, \quad \frac{d^2\varphi x'}{dx'^2} = \frac{d^2Fx'}{dx'^2},$$

these curves have with each other a contact of the second order; if, besides these equations, we have also

$$\frac{d^3\varphi x'}{dx'^3} = \frac{d^3Fx'}{dx'^3},$$

the contact will be of the third order, and so on.

149. The equation to the circle, which is

$$(y-\beta)^2+(x-\alpha)^2=\gamma^2,$$

contains three constants; and we may therefore determine the circle which has a contact of the second order with any curve

Fig. 25. MN (fig. 25), of which we know the equation. For this purpose, let x' and y' be the coordinates of the point M in the circumference of the circle; the value of y' will then be given by the equation

$$(y'-\beta)^2+(x'-\alpha)^2=\gamma^2,$$

and must replace Fx' in the equations of contact, which are

$$\varphi x'=Fx',\ \frac{d\varphi x'}{dx'}=\frac{dFx'}{dx'},\ \frac{d^2\varphi x'}{dx'^2}=\frac{d^2Fx'}{dx'^2}.$$

If at the same time we take x and y for the coordinates of the curve $y=\varphi x$, at the point of contact, the preceding equations will become

$$y=y',\ \frac{dy}{dx}=\frac{dy'}{dx'},\ \frac{d^2y}{dx^2}=\frac{d^2y'}{dx'^2}\ \ldots.\ (74),$$

and in these we must substitute for the quantities y', $\frac{dy'}{dx'}$, and $\frac{d^2y'}{dx'^2}$, their values derived from the equation (73), and its successive differentials, which are

$$(y'-\beta)\ \frac{dy'}{dx'},+x'-\alpha=0\ \ldots.\ (75),$$

$$(y'-\beta)\ \frac{d^2y'}{dx'^2}+\frac{dy'^2}{dx'^2}+1=0\ \ldots.\ (76).$$

But the substituting in the equations (74) the values of y', $\frac{dy'}{dx'}, \frac{d^2y'}{dx'^2}$, given by the equations (73), (75), (76), will be the same thing with eliminating these quantities betwixt the equations (73), (74), (75), and (76), which will be done by effacing the accents in the equations (73), (75), and (76); observing at the same time that when

$$y=y',\ \text{we have}\ x=x'.$$

Suppressing the accents, therefore, we shall find

$$(y-\beta)^2+(x-\alpha)^2=\gamma^2 \quad \ldots \quad (77),$$

$$(y-\beta)\frac{dy}{dx}+x-\alpha=0 \quad \ldots \quad (78),$$

$$(y-\beta)\frac{d^2y}{dx^2}+\frac{dy^2}{dx^2}+1=0 \quad \ldots \quad (79):$$

from the last of which equations we deduce

$$y-\beta=-\frac{\left(1+\frac{dy^2}{dx^2}\right)}{\frac{d^2y}{dx^2}} \quad \ldots \quad (80);$$

and putting this value in the equation (78) we obtain

$$x-\alpha=\frac{\left(1+\frac{dy^2}{dx^2}\right)}{\frac{d^2y}{dx^2}}\frac{dy}{dx} \quad \ldots \quad (81).$$

If in the equation (77), we substitute these values of $y-\beta$ and $x-\alpha$, it becomes

$$\frac{\left(1+\frac{dy^2}{dx^2}\right)^2}{\left(\frac{d^2y}{dx^2}\right)^2}+\frac{\left(1+\frac{dy^2}{dx^2}\right)^2}{\left(\frac{d^2y}{dx^2}\right)^2}\frac{dy^2}{dx^2}=\gamma^2,$$

and, adding the numerators which have a common factor, we shall have

$$\frac{\left(1+\frac{dy^2}{dx^2}\right)^2\left(1+\frac{dy^2}{dx^2}\right)}{\left(\frac{d^2y}{dx^2}\right)^2}=\gamma^2,$$

an equation which reduces itself to

$$\frac{\left(1+\frac{dy^2}{dx^2}\right)^3}{\left(\frac{d^2y}{dx^2}\right)^2}=\gamma^2.$$

and, by extracting the square root, gives

$$\pm\frac{\left(1+\frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}=\gamma.$$

150. The double sign refers to the position of γ: if the curve be concave to the axis of x, then $\frac{d^2y}{dx^2}$ will be negative; and in order that γ may then result positively, we must take γ with the negative sign, and write

$$\gamma=-\frac{\left(1+\frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} \quad . \quad . \qquad (82):$$

for the curve being concave to the axis of the abscissæ $\frac{d^2y}{dx^2}$ appears as a negative quantity, and therefore, when substituted in equation (82), will render the value of γ positive.

151. The circle which we have just been considering has received the name of the *osculating circle*, and its radius that of the *radius of curvature*; in order, therefore, to obtain the radius of curvature, we require only to have the equation of the curve, from which to deduce the differential coefficients that are to be substituted in the formula (82).

If the curve ought to be convex to the axis of x, the positive sign must be prefixed to the value of γ.

152. The value of γ is sometimes written thus,

$$\gamma=-\frac{(dx^2+dy)^{\frac{3}{2}}}{dx\ d^2y};$$

a form which is readily deduced from the equation (82); for by reducing the two terms within the brackets to a common denominator, and observing that $(dx^2)^{\frac{3}{2}}$ is dx^3, we obtain

$$\gamma = -\frac{(dx^2+dy^2)^{\frac{3}{2}}}{dx^3.\frac{d^2y}{dx^2}} = -\frac{(dx^2+dy^2)^{\frac{3}{2}}}{dx\,d^2y}.$$

153. As an application of the formula (82), let it be proposed to find the radius of curvature of the parabola NAM (fig. 26), the equation to which is $x^2=my$. Fig. 26.

This equation gives

$$2xdx = mdy,\ \frac{dy}{dx}=\frac{2x}{m},\ \frac{d^2y}{dx^2}=\frac{2}{m}:$$

therefore

$$\gamma = \frac{\left(1+\frac{4x^2}{m^2}\right)^{\frac{3}{2}}}{\frac{2}{m}} = \frac{\left[\frac{4}{m^2}\left(\frac{m^2}{4}+x^2\right)\right]^{\frac{3}{2}}}{\frac{2}{m}},$$

and raising the two factors to the power $\frac{3}{2}$, we have

$$\gamma = \frac{8}{m^3}\cdot\frac{\left(\frac{m^2}{4}+x^2\right)^{\frac{3}{2}}}{\frac{2}{m}} = \frac{\left(\frac{m^2}{4}+x^2\right)^{\frac{3}{2}}}{\frac{m^2}{4}}\ \ldots (83);$$

but the normal to the parabola has for its expression . . . . $\left(\frac{m^2}{4}+x^2\right)^{\frac{1}{2}}$; and we see, therefore, that *the radius of curvature of the parabola is equal to the cube of the normal, divided by the square of the semi-parameter.*

154. The osculating circle will serve to measure the curvature of the curve at any point M (fig. 25); for if at that point M we describe, with the radius of curvature, an exceedingly small circular arc ML, that arc may be considered as the arc of the curve itself, from which it separates but in a very slight degree. Now the greater be the curvature of the arc ML, the less is its radius; and it follows, therefore, that from the decrease or increase of the radius of curvature, we

may determine the increase or decrease of the curvature of the curve.

If, for example, we examine the equation (83), which gives the radius of curvature of the parabola, we see that at the vertex of the curve, where $x=0$, $\gamma=\frac{m}{2}$; but that when x is successively increased, γ increases; which intimates, therefore, that the curvature of the parabola goes on continually decreasing, as we retire from the vertex.

155. $\frac{dy}{dx}$ expressing the trigonometrical tangent of the angle which the tangent at M (27) makes with the axis of x, the equation of the normal made to pass through a point whose coordinates are α and β, will be

$$y-\beta=-\frac{dx}{dy}(x-\alpha),$$

and this equation being the same with the equation (78) in which α and β are the coordinates of the centre of the osculating circle, we see that the radius of that circle is a normal to the curve.

156. If now, through all the points of a curve MM′M″, &c. (fig. 28), we draw the radii of curvature MO, M′O′, M″ O″, &c., we shall construct a series of points O, O′, O″, &c.; which points being all subject to a certain law *, we may give to their system the name of curve; though we cannot yet say any thing of the nature of this new curve, which we call the *evolute* of the curve MM′M″; this latter curve, considered relatively to the evolute, being called the *involute*.

157. If now we pass from one point of the evolute to an-

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* This law is implicitly contained in the equation of the curve MM′M″; for that curve being given, the position of the points O, O′, O″, &c. results from it.

other, not only will x and y vary, but α, β, and γ will also vary at the same time; for since α and β are, generally, the coordinates of the centre of the osculating circle, and the evolute is formed by the system of those centres, it follows that α and β are the coordinates of the evolute, and therefore coordinates which must vary for different points of the curve. It is the same with γ, which is the radius of the osculating circle, and represents the distance of any point of the evolute from that point of the involute whence γ is drawn; and consequently, by differentiating the equation (78), in respect of all the letters *, and dividing by dx, we shall obtain

$$(y-\beta)\frac{d^2y}{dx^2}+\frac{d^2y}{dx^2}-\frac{dy}{dx}\cdot\frac{d\beta}{dx}+1-\frac{d\alpha}{dx}=0\,;$$

and subtracting the equation (79) from this, there remains

$$-\frac{dy}{dx}\cdot\frac{d\beta}{dx}-\frac{d\alpha}{dx}=0,$$

whence we find

$$\frac{dy}{dx}=-\frac{\frac{d\alpha}{dx}}{\frac{d\beta}{dx}}=-\frac{d\alpha}{dx}\times\frac{1}{\frac{d\beta}{dx}}.$$

But, art. 67,

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* We cannot differentiate otherwise the equation

$$(y-\beta)^2+(x-\alpha)^2=\gamma^2$$

and its derivates; and yet we appear to have done so when from the equation (73) we have deduced the equations (75) and (76); to which it may be answered, that having two arbitrary constants in the equation (73), we have determined them on the condition that the functions represented by the first sides of the equations (75) and (76) should be each equal to 0; without this, we should have had no right to conclude that because the equation (73) holds good, the equations (75) and (76) must hold good also.

$$\frac{1}{\frac{d\beta}{dx}}=\frac{dx}{d\beta},$$

therefore,

$$\frac{dy}{dx}=-\frac{d\alpha}{dx}\times\frac{dx}{d\beta},$$

and, consequently, art. 24,

$$\frac{dy}{dx}=-\frac{d\alpha}{d\beta};$$

which value of $\frac{dy}{dx}$ being substituted in the equation (78), we shall obtain

$$y-\beta=\frac{d\beta}{d\alpha}(x-\alpha)\ \ldots\ (84).$$

158. We saw, art. 155, that the equation $y-\beta=-\frac{dx}{dy}(x-\alpha)$ was that of the radius of curvature, passing through the point whose coordinates are x and y; and it will therefore be always the equation of the same radius, when $-\frac{dx}{dy}$ is replaced by $\frac{d\beta}{d\alpha}$. But the equation (84) is also that of a tangent drawn at the point of the evolute, whose co-ordinates are α and β*; and the radius of curvature is therefore a tangent to the evolute.

159. Since in the following demonstration we shall have to employ the differential of an arc of a curve, we will proceed to find that differential.

Suppose that an abscissa AP$=x$ (fig. 31) receives an increment $PP'=h$; if then we draw MO parallel to the axis of x, we shall have evidently

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* We may observe, that α and β being generally the coordinates of any point in the evolute, the equation of the evolute will be $\beta=f\alpha$; and therefore $\frac{d\beta}{d\alpha}$ expresses the tangent of the angle which the tangent at the point α, β, makes with the axis of the abscissæ.

$$\text{chord } MM'=\sqrt{MO^2+M'O^2}=\sqrt{h^2+M'O^2};$$

but

$$M'O=f(x+h)-fx=\frac{dy}{dx}h+\frac{d^2y}{dx^2}\frac{h^2}{1.2}+\text{ \&c.};$$

substituting which value in the expression for MM′, and representing the coefficients of h^3, h^4, &c. by A, B, &c. we shall have

$$MM'=\sqrt{h^2+\frac{dy^2}{dx^2}h^2+Ah^3+Bh^4+\text{ \&c.}},$$

or

$$MM'=\sqrt{h^2\left(1+\frac{dy^2}{dx^2}\right)+Ah^3+Bh^4+\text{ \&c.}},$$

and therefore

$$\frac{MM'}{h}=\sqrt{1+\frac{dy^2}{dx^2}+Ah+Bh^2+\text{ \&c.}}$$

In the case of the limit we have $h=0$, and the chord coincides with the arc which we will represent by s; so that we shall have

$$\frac{ds}{dx}=\sqrt{1+\frac{dy^2}{dx^2}};$$

whence, multiplying by dx, we deduce

$$ds=\sqrt{dx^2+dy^2}.$$

160. For the evolute the coordinates are α and β; and for it, therefore, we shall have in like manner

$$ds=\sqrt{d\alpha^2+d\beta^2}.$$

161. If now we differentiate the equation (77) in respect of all the letters, we shall find

$$(y-\beta)\,(dy-d\beta)+(x-\alpha)\,(dx-d\alpha)=\gamma d\gamma,$$

and equation (78) gives us

$$(y-\beta)\,dy+(x-\alpha)\,dx=0,$$

which equation being subtracted from the preceding one, we have remaining

$$-(y-\beta)\,d\beta-(x-\alpha)\,d\alpha=\gamma d\gamma \;\ldots\; (85).$$

Substituting in this equation (85) and in the equation (77) the value of $y-\beta$, given by the equation (84), we shall find the two equations

$$-\frac{d\beta^2}{d\alpha}\,(x-\alpha)-(x-\alpha)d\alpha=\gamma d\gamma,$$

$$\frac{d\beta^2}{d\alpha^2}\,(x-\alpha)^2+(x-\alpha)^2=\gamma^2\,;$$

which, putting $x-\alpha$ as a common factor, and extracting the square root of the second, become

$$-(x-\alpha)\,.\,\frac{d\beta^2+d\alpha^2}{d\alpha}=\gamma d\gamma,$$

$$(x-\alpha)\frac{\sqrt{d\alpha^2+d\beta^2}}{d\alpha}=\gamma,$$

and dividing the first of these equations by the second, we obtain

$$d\gamma=-\sqrt{d\beta^2+d\alpha^2}.$$

But we have seen, art. 160, that, representing by s an arc of the evolute, we have

$$ds=\sqrt{d\beta^2+d\alpha^2}\,;$$

comparing which equation with the preceding one, we deduce

$$d\gamma=-ds, \text{ or } d(\gamma+s)=0\,;$$

and since every function whose differential is 0 must be constant, we have $\gamma+s=0$; and therefore if the radius of curvature increase, the arc s must diminish, and by the same quantity; a relation which we express by saying that *the radius of curvature varies by the same differences as the evolute.*

162. Let (fig. 29) $MO=\gamma$, $OB=s$, $M'O'=\gamma'$, $O'B=s'$; we have then for the radius of curvature MO,

$$\gamma+s=constant,$$

or

$$MO+\text{arc}\,OB=constant \;\ldots\; (86).$$

The radius of curvature M′O′ gives rise, similarly, to the equation

$$\gamma'+s'=constant,$$

or

$$M'O'+\text{arc}\,O'B=constant \;\ldots\; (87);$$

and since the second sides of these equations (86) and (87) represent the same constant, we derive from them,

$$M'O'+\text{arc}\,O'B=MO+\text{arc}\,OB,$$

when consequently,

$$M'O'-MO=\text{arc}\,OB-\text{arc}\,O'B=\text{arc}\,OO',$$

which shows us that the *difference of two radii of curvature is equal to the arc comprehended between them.*

163. It follows from this, that if on the evolute OB (fig. 29) we wrap a string OBM, which, being of course a tangent to the evolute at the point B where it leaves it, has its extremity in the point M of the involute C, when we unwrap this string, keeping it constantly stretched, its extremity M will trace out in its course the involute MC; for supposing that in the course of its motion it has reached the position O′M, it will be increased by OO′, and will consequently be equal in length to the radius of curvature passing through the point O′, whence the extremity M′ of the string will be still in the involute MC.

164. The equation of the evolute is determined in the following manner: 1°. we deduce from the equation of the curve, the values of y, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$; 2°. we substitute these values in the

equations (78) and (79), when we shall get two new equations, which are functions of x alone; 3°. eliminating x between these equations, we arrive at an equation betwixt α and β; this equation will be that of the evolute.

165. To determine by this process the evolute of the parabola, whose equation is $x^2=my$; we have, by differentiating,

$$2xdx=mdy,$$

and consequently

$$\frac{dy}{dx}=\frac{2x}{m},\ \frac{d^2y}{dx^2}=\frac{2}{m};$$

which values of y, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, being substituted in the equations (78) and (79), they become

$$\left(\frac{x^2}{m}-\beta\right)\frac{2x}{m}+x-\alpha=0 \quad .\ .\ .\ .\ (88),$$

$$\left(\frac{x^2}{m}-\beta\right)\frac{2}{m}+\frac{4x^2}{m^2}+1=0 \quad .\ .\ .\ (89),$$

and subtracting the equation (88) from the equation (89), multiplied by x, we shall obtain

$$\alpha+\frac{4x^3}{m^2}=0 \quad .\ .\ .\ .\ (90).$$

On the other hand, the equation (89) multiplied by m^2, and reduced, gives us

$$6x^2-2m\beta+m^2=0,$$

whence we find

$$\beta=\frac{3x^2}{m}+\frac{m}{2} \quad .\ .\ .\ .\ (91);$$

and eliminating x betwixt the equations (90) and (91), we shall have the equation of the evolute.

But before we perform this operation, we may observe that at the origin, where $x=0$, the equations (90) and (91) are reduced to $\alpha=0$, $\beta=\frac{m}{2}$; and taking, therefore, $AB=\frac{m}{2}$ (fig. 32), we have the point B of the evolute; we see, then, from the equation (91), that giving to x values positive or negative, β goes on increasing according as these values increase; whence it follows that the evolute is formed of two branches BC and BD.

166. To eliminate x betwixt the equations (90) and (91), the first, raised to the square, gives

$$x^6=\alpha^2\frac{m^4}{16};$$

from the second we deduce

$$x^2=\left(\beta-\frac{m}{2}\right)\frac{m}{3};$$

the two sides of which being cubed, we find

$$x^6=\left(\beta-\frac{m}{2}\right)^3\frac{m^3}{27},$$

and equating these two values of x^6, and dividing by m^3, we obtain

$$\alpha^2.\frac{m}{16}=\left(\beta-\frac{m}{2}\right)^3\frac{1}{27}.$$

Let $\beta-\frac{m}{2}=\beta'$, multiply each side by 27, and make . . . . $\frac{27}{16}m=n$; then the equation becomes

$$\beta'^3=n\alpha^2\ (*),$$

which is the equation of the evolute.

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[*] It is easy to prove that the branches BC, BD have their convexities opposed to each other; for by differentiating the equation $\beta'^3=n\alpha^2$, or . . . . .

Fig. 22. By the assumption of $\beta-\frac{m}{2}=\beta'$, the origin is transformed to the point B, whose coordinates are 0 and $\beta-\frac{m}{2}$.

167. An osculating curve may be situated in two different ways in respect of the curve with which it is in contact: 1°. it may have its two branches, both of them above the
Fig. 33, 34. curve, as in fig. 33, or both of them below, as in fig. 34; in which case the osculate will only touch the curve: 2°. the osculate may have one branch above and the other below the
Fig. 35. curve, as in fig. 35; and in this case the osculate will cut the curve in the point M.

168. We will proceed to show, that the osculating circle
Fig. 36. (fig. 36) cuts the curve.

For the same abscissa $x+h$
let Y be the ordinate of the curve,
Y′ be the ordinate of the osculate;

we have then

$$\left.\begin{array}{l} Y=\varphi(x+h)=\varphi x+Ah+Bh^2+Ch^3+\&c. \\ Y'=F(x+h)=Fx+A'h+B'h^2+C'h^3+\&c. \end{array}\right\}\ldots(92);$$

and since the circle is an osculate of the second order, the three first terms of these developments will be the same; whence the difference of the ordinates, corresponding to $x+h$, will be

$$(C-C')h^3+, \&c. \ldots (93).$$

Suppose, now, that the abscissa becomes $x-h$; we must then change h into $-h$ in the difference of the ordinates, which will become

$$-(C-C')h^3+, \&c. \ldots (94);$$

$\beta'=n^{\frac{1}{3}}\alpha^{\frac{2}{3}}$, we find $\frac{d^2\beta'}{d\alpha^2}=-\frac{2}{9}n^{\frac{1}{3}}\alpha^{-\frac{4}{3}}=-\frac{2}{9}\sqrt[3]{\frac{n}{\alpha^4}}$, a negative value for either a positive or negative value of α, which proves that each branch is concave to the axis of x.

and since, by taking h sufficiently small, the first term of the series (93) and (94) may be made greater than the sum of all the other terms, it follows that the difference of the ordinates will change its sign, when the abscissa, instead of being $x+h$, shall become $x-h$. Hence, taking (fig. 36) $PP''=PP'=h$, if the difference of the ordinates corresponding to $x+h$ be a positive quantity, i. e. if the ordinate $P'M'$ of the curve be greater than $P'N'$, the ordinate $P''N''$ of the osculate will be greater than the ordinate $P''M''$ of the curve; whence we conclude that the osculate is on one side above the curve, and on the other below it, and consequently cuts it. What we have said of the circle, which is an osculate of the second order, will apply to every osculate of an even order.

169. If the osculate be of an odd order, it will only touch the curve, instead of cutting it; as is evident from the preceding demonstration.

170. We will now give the theorem promised art. 138, respecting *multiple points*. If the curves which meet in one of those points have a common tangent, the equation to which may be represented by $y=ax+b$, we must change Fx into $ax+b$ in the second of the equations (92), which will give $\frac{dFx}{dx}$ or $A'=a$, and all the rest of the coefficients in that equation will vanish; also the tangent being an osculate of the first order, $\phi x+Ah$ will be equal to $Fx+A'h$, which will reduce the difference of the equations (92) to

$$Y-Y'=Bh^2+Ch^3+, \&c.$$

Now this difference ought to have two values QM and QM' (fig. 30), and therefore one of the differential coefficients represented by B, C, &c., must have two values. Let $\frac{d^n\phi x}{dx^n}$ be this coefficient; we have seen already (art. 143), that if we take the successive differentials of the equation $Pdx+Qdy=0$, after each differentiation, the term Q will continue a factor of the differential of the highest order of y; so that the differential of the nth order of the proposed function may be represented by $Q\frac{d^ny}{dx^n}+k=0$; and since $\frac{d^ny}{dx^n}$ must have two values, we might prove, as in art. 137, that $Q=0$. This value of Q will reduce that of P to 0; and it follows, therefore, that the equation $\frac{dy}{dx}=-\frac{P}{Q}$ will give $\frac{dy}{dx}=\frac{0}{0}$.

The development of functions of two variables.

171. When in a function, u, of two independent variables, x and y, we change x into $x+h$, and y into $y+k$, Taylor's theorem will give us the means of developing this function: for suppose that first we substitute $x+h$ in place of x, we shall have then, by that theorem,

$$f(x+h,y)=u+\frac{du}{dx}h+\frac{d^2u}{dx^2}\frac{h^2}{1.2}+\frac{d^3u}{dx^3}\cdot\frac{h^3}{1.2.3}+\&c.\ \dots\ (95);$$

in which series y can be contained only in the functions u, $\frac{du}{dx}$, $\frac{d^2u}{dx^2}$, &c. Changing, therefore, y into $y+k$ in these functions, we must replace, in the equation (95),

$$u \text{ by } u+\frac{du}{dy}k+\frac{d^2u}{dy^2}\frac{k^2}{1.2}+\frac{d^3u}{dy^3}\frac{k^3}{2.3}+\&c.;$$

$$\frac{du}{dx} \text{ by } \frac{du}{dx}+\frac{d.\frac{du}{dx}}{dy}k+\frac{d^2\frac{du}{dx}}{dy^2}\cdot\frac{k^2}{1.2}+\frac{d^3\frac{du}{dx}}{dy^3}\frac{k^3}{2.3}+\&c.;$$

$$\frac{d^2u}{dx^2} \text{ by } \frac{d^2u}{dx^2}+\frac{d.\frac{d^2u}{dx^2}}{dy}k+\frac{d^2.\frac{d^2u}{dx^2}}{dy^2}\frac{k^2}{2}+\frac{d^3.\frac{d^2u}{dx^2}}{dy^3}\frac{k^3}{2\,3}+\&c.;$$

&c., &c., &c., &c., &c.;

and forming as many lines as there are terms in the equation (95), we shall obtain

$$\left.\begin{aligned} f(x+h,\ y+k)=u&+\frac{du}{dy}k+\frac{d^2u}{dy^2}\frac{k^2}{2}+\&c.\\ &+\frac{du}{dx}h+\frac{d.\frac{du}{dx}}{dy}hk+\&c.\\ &+\frac{d^2u}{dx^2}\frac{h^2}{2}+\&c.\\ &+\&c.\end{aligned}\right\}\ \dots\dots\dots\ (96).$$

172. If we had made these substitutions in an inverse order, we should have found, first, by changing y into $y+k$,

$$f(x, y+k)=u+\frac{du}{dy}k+\frac{d^2u}{dy^2}\frac{k^2}{2}+\frac{d^3u}{dy^3}\frac{k^3}{2.3}+\&c.;$$

and putting then in each term $x+h$ in place of x, we should have arrived at this development,

$$\left.\begin{aligned} f(y+k, x+h)=u+\frac{du}{dx}h+\frac{d^2u}{dx^2}\frac{h^2}{2}+\&c. \\ +\frac{du}{dy}k+\frac{d.\frac{du}{dy}}{dx}hk+\&c. \\ +\frac{d^2u}{dy^2}\frac{k^2}{2}+\&c. \\ +\&c. \end{aligned}\right\}\ldots\ldots\ldots(97).$$

The order in which we make these substitutions being arbitrary, (since in putting $x+h$ wherever x enters, and $y+k$ wherever y enters, these operations cannot affect each other), it follows, that the two developments (96) and (97) must be identical, and consequently, the terms affected by the same products of h and k have the same values.

Equating therefore the terms which are multiplied by hk, we shall obtain

$$\frac{d.\frac{du}{dx}}{dy}=\frac{d.\frac{du}{dy}}{dx}, \text{ or } \frac{d^2u}{dxdy}=\frac{d^2u}{dydx};$$

an equation which shows us that, in taking the second differential of the product of two variables, the order of the differentiations is arbitrary.

The same thing may be proved for the differential coefficients of higher orders, by equating the differential coefficients of the other terms of the equations (96) and (97).

Maxima and minima in functions of two variables.

173. We have seen, art. 171, that if in a function of two independent variables x and y, we replace x by $x+h$, and y by $y+k$, the development of $f(x+h,\ y+k)$ will be given by the equation (96); in which equation, if we represent $f(x+h,\ y+k)$ by U, k by mh, and $\frac{d.\frac{du}{dx}}{dy}$ by $\frac{d^2u}{dxdy}$, we shall have

$$\mathrm{U}=u+h\left(\frac{du}{dy}m+\frac{du}{dx}\right)+\frac{h^2}{2}\left(\frac{d^2u}{dy^2}m^2+2\frac{d^2u}{dxdy}m+\frac{d^2u}{dx^2}\right)$$
$$+\text{ terms in } h^3, h^4, \text{\&c.} \ldots . (98).$$

Now, in order that u may be a maximum or minimum, it is necessary that, whatever be the values given to the increments h and k, U should be always less or always greater than u; but this is only possible when the term $h\left(\frac{du}{dy}m+\frac{du}{dx}\right)$ is evanescent; for if it be not, then, by taking a proper value of h, this term may be rendered greater than the sum of all the following terms, when by taking h successively positive and negative, we should make U in one case greater, and in the other less, than u; hence, in order that u may be a maximum or a minimum, we must have

$$h\left(\frac{du}{dy}m+\frac{du}{dx}\right)=0,$$

or

$$\frac{du}{dy}m+\frac{du}{dx}=0.$$

Since also k is arbitrary, m must be so likewise; and, consequently, this equation must hold good, whatever be the value of m, which requires that the equation resolve itself into these two:

$$\frac{du}{dy}=0,\quad \frac{du}{dx}=0.$$

174. We must now see what it is that distinguishes the maximum from the minimum; for which purpose we must observe, that since the term in h vanishes, it is the term in h^2, which will decide the sign of the sum of all the terms following u, and that, consequently, the term in h^2, if it do not vanish, must not, for the different values of h and k, result at one time positive, at another negative, or otherwise U might be in one case less, and in the other greater than u. We will therefore proceed to investigate the conditions

necessary that this term in h^2 may always have the same sign, whatever be the values we give to h and k; and, with this view, we will represent the term by

$$\frac{h^2}{2}(Am^2+2Bm+C);$$

making A a common factor, this will become

$$\frac{Ah^2}{2}\left(m^2+2\frac{B}{A}m+\frac{C}{A}\right)\ldots\ldots\ (99),$$

and adding and subtracting the same quantity $\frac{B^2}{A^2}$, the expression (99) may be written thus,

$$\frac{Ah^2}{2}\left[\left(m+\frac{B}{A}\right)^2+\frac{C}{A}-\frac{B^2}{A^2}\right]\ldots\ (100),$$

which, if C and A be of the same sign, and $\frac{C}{A}$ be $>\frac{B^2}{A^2}$, i. e. $AC>B^2$, will always have the same sign as A; for then the quantity multiplied by $\frac{Ah^2}{2}$ will be essentially positive, and the sign of the expression (100) will depend on that of A; so that we shall have a maximum or a minimum, accordingly as A shall be negative or positive, i. e. according to the sign of $\frac{d^2u}{dy^2}$, which will be the same with that of $\frac{d^2u}{dx^2}$, for, by hypothesis, C and A are to have both the same sign.

On the transformation of rectangular coordinates to polar.

175. Let BDC (fig. 79) be a curve, in which we have determined a point M, in position, by means of the rectangular coordinates $AP=x$, $PM=y$; this point may be equally determined, if we have given the angle MAC, and the radius vector AM: but since we generally measure angles by arcs, we will substitute for the angle MAC the circular arc *mo*, described with a radius unity; and thus, representing the arc *mo* by θ, and the radius vector AM by u, we may substitute the system of polar coordinates θ and u, instead of that of the rectangular coordinates $AP=x$, and $PM=y$. Fig. 79.

176. The origin of the abscissæ is sometimes placed elsewhere than in *o*; for the point M will be equally determined, if, having taken o' for the origin, we have given the arc $o'm$, and the radius vector AM. In this case we may represent $o'm$ by θ', and then the abscissæ, reckoned from the origin o, will differ from the abscissæ reckoned from o', by a constant quantity oo'; and there will exist between them the following equation:

$$\theta = \theta - oo'.$$

Since, by means of this equation, we can always change the origin in any manner required, we will, for greater simplicity, suppose the origin in o.

177. Representing now by $F(x, y) = 0$, the equation in which we wish to change the rectangular coordinates $AP = x$ and $PM = y$ into polar coordinates $om = \theta$, and $AM = u$; and investigating the relations that exist betwixt these coordinates, we see at once, that

$$AP = AM \cos MAP, \quad PM = AM . \sin MAP,$$

or

$$x = u \cos \theta, \qquad y = u . \sin \theta . \; . \; . \; . \; (101);$$

and we have only therefore to substitute these values in the equation represented by $F(x, y) = 0$, in order to obtain the equation in respect of the polar coordinates.

Fig. 80. 178. If the origin of the rectangular coordinates x and y be not in the centre A of the curve (fig. 80); let x', y', be the coordinates reckoned from the origin A′, and a and b the coordinates of the centre A, reckoned from that origin; then we shall have

$$AP = A'Q - A'B, \quad MP = MQ - AB,$$

or

$$x = x' - a, \qquad y = y' - b;$$

which values we must substitute in the preceding formulæ.

On the transformation of polar coordinates into rectangular, and the determination of the differential of the arc of a polar curve.

179. The equation in respect of polar coordinates being re-

presented by $F(\theta, u)=0$, we see at once (fig. 79) that we may Fig. 79. replace u by its value derived from the equation

$$AM^2 = AP^2 + PM^2,$$

or

$$u^2 = x^2 + y^2 \; \ldots . \; (102).$$

In regard to θ, the equations (101), divided the one by the other, give us

$$\frac{y}{x} = \frac{\sin\theta}{\cos\theta} = \tan\theta,$$

whence we deduce

$$\theta = \operatorname{arc}\left(\tan = \frac{y}{x}\right) = \tan^{-1} \cdot \frac{x}{y}.$$

These values of θ and u being substituted in the equation $F(\theta, u)=0$, we obtain

$$F\left(\tan^{-1} . \frac{x}{y}, \; \sqrt{x^2+y^2}\right) = 0 \; . \; . \; . \; (103),$$

and so arrive at an equation between x and y, involving a transcendental quantity $\tan^{-1} \frac{x}{y}$.

180. We may also obtain, between x and y, an equation, which shall not contain the transcendental arc $\tan^{-1} \frac{x}{y}$, but which will involve differentials; and, for this purpose, we might at once differentiate the equation represented by formula (103); but the method generally adopted for arriving at this end is the following: representing always by $F(\theta, u) = 0$, the equation which it is required to transform into a function of the rectangular coordinates x, y, we have seen, art. 179, that the value of u may be expressed in terms of x and y, without any transcendental, but that the same cannot be done in respect to θ; on which account, therefore, we eliminate θ between the equation $F(\theta, u)=0$, and its differential, which we will represent by $F(\theta, ud\theta, du)=0$; this process will, in

fact, introduce into the result the differentials $d\theta$ and du; but we shall see that these differentials may be expressed in functions of the variables x, y, dx and dy.

For, first, the equations (101) give us

$$\cos\theta=\frac{x}{u},\quad \sin\theta=\frac{y}{u}\;.\,.\,.\,.\,.\;(104);$$

dividing one of these equations by the other, we obtain

$$\frac{\sin\theta}{\cos\theta}=\frac{x}{y}\text{ or }\tan\theta=\frac{x}{y};$$

differentiating, there results

$$\frac{d\theta}{\cos^2\theta}=\frac{xdy-ydx}{x^2};$$

and replacing $\frac{1}{\cos^2\theta}$ by its value, derived from the first of the equations (104), and suppressing the common factor x, we find

$$u^2d\theta=xdy-ydx;$$

whence, consequently,

$$d\theta=\frac{xdy-ydx}{u^2}\;.\,.\,.\,.\,.\;(105);$$

and putting for u its value, this equation becomes

$$d\theta=\frac{xdy-ydx}{x^2+y^2}.$$

The differential of the other variable is found still more easily, for the equation (102) gives us

$$u=\sqrt{x^2+y^2};$$

which being differentiated, we have

$$du=\frac{xdx+ydy}{\sqrt{x^2+y^2}};$$

and, by means of these values of $d\theta$, du, and u, we shall change the equation obtained by the elimination of θ into another involving only x, y, dy, and dx; and which, consequently, will be the equation belonging to the rectangular coordinates.

181. We have seen, art. 159, that the differential of an arc z, referred to rectangular coordinates, has for its expression,

$$dz=\sqrt{dx^2+dy^2} \quad \ldots\ldots \quad (106).$$

It may be proposed to find the differential of the same arc, when the coordinates are polar ones; and, in this case, we must substitute in the equation (106) the values of dx and dy, derived from the equations

$$x=u.\cos\theta,\ y=u.\sin\theta.$$

Now, by differentiating these equations, we shall find

$$dx=-u\sin\theta.d\theta+\cos\theta du,$$
$$dy=\quad u\cos\theta\, d\theta+\sin\theta du;$$

whence, squaring these last equations, and reducing them by means of the formula

$$\sin^2\theta+\cos^2\theta=1,$$

we shall obtain

$$dz=\sqrt{u^2d\theta^2+du^2};$$

which is the differential of the arc in a function of the polar coordinates.

On subtangents and subnormals, tangents and normals, to polar curves.

182. We know that, in curves referred to rectangular coordinates, the subtangent A*t* (fig. 81) is the line comprised between the foot P of the ordinate and the point *t*, in which a perpendicular, A*t*, to that ordinate meets the tangent: retaining the same definition for polar curves, in which the ordinate is no longer PM, but the radius vector AM, the subtangent will be then the line AT, drawn perpendicular to AM, and comprised between the point A and the point T, in which that perpendicular cuts the tangent. The subtangent Fig. 81.

therefore has, in polar curves, a position different to that it has in curves referred to rectangular coordinates, since in these the subtangent is always measured along the axis of the abscissæ, whereas in polar curves, in which that axis no longer exists, the subtangent varies its position at every point of the curve.

Fig. 82. 183. We will now determine the analytical expression for the subtangent of polar curves; for which purpose, let AM and AM' be two radii vectores; from the point M draw the perpendicular MP to the radius vector AM', and to that perpendicular draw the parallel AT; then the similar triangles ATM', PMM', will give us the proportion

$$PM' : PM :: AM' : AT;$$

whence we deduce

$$AT = \frac{PM \times AM'}{PM'};$$

and observing that PM' is a side of the right-angled triangle PMM', this value of AT becomes

$$AT = \frac{PM \times AM'}{\sqrt{MM'^2 - PM^2}}.$$

In the case of the limit, AM' is equal to AM, i. e. to u, PM coincides with the arc MN, the chord MM' with the arc MM, and AT becomes the subtangent. For the limit, therefore, we have only to determine the expressions for M'M and MN; the first of which being then the differential of the arc of the curve, we have, art. 181,

$$MM' = \sqrt{u^2 d\theta^2 + du^2};$$

in regard to MN, the sectors ARR' and AMN give us the proportion

$$AR : RR' :: AM : MN,$$

or

$$1 : RR' :: u : MN,$$

and therefore $MN = u \, . \, RR'$, a quantity which, in the case of the limit, reduces itself to $ud\vartheta$.

Putting these values of MN and M'M in that of AT, changing AM' into u, and reducing, we shall find

$$AT = \frac{u^2 d\theta}{du},$$

which is the expression for the subtangent.

184. To determine the subnormal, we must observe that the normal SM being perpendicular to the tangent, the ordinate AM (fig. 81) must be a mean proportional betwixt the subtangent and the subnormal; whence, consequently, we have

$$AT : AM : . \; AM : \mathit{subnormal},$$

or

$$\frac{u^2 d\theta}{du} : u : : u : \mathit{subnormal};$$

and therefore

$$\mathit{subnormal} = \frac{d\vartheta}{du}.$$

In regard to the normal and tangent, the right-angled triangles MAS, MAT give

$$MS = \sqrt{MA^2 + AS^2}, \quad MT = \sqrt{MA^2 + AT^2};$$

and substituting in these equations the values of MA, AS, and AT, we shall find

$$\mathit{normal} = \sqrt{u^2 + \frac{du^2}{d\theta^2}}, \quad \mathit{tangent} = u\sqrt{1 + u^2\frac{d\theta^2}{du^2}}.$$

185. To find the analytical expression of the sector in polar curves, the triangle AM'M (fig. 82) gives us Fig. 82.

$$\mathit{area}\ AM'M = \frac{AM' \times PM}{2};$$

in the case of the limit, the area of the triangle AM'M (fig. 82) becomes that of an elementary sector, the perpendicular

PM may be replaced by the arc MN, which we have found equal to $ud\theta$, and AM′ by u; when, making these substitutions, we shall find

$$\textit{area of the elementary sector}=\frac{u^2d\theta}{2}.$$

The elementary sector may also be expressed in a function of the rectangular coordinates, for, by putting in this equation the values of u and $d\theta$, given by the equations 102 and 105, it becomes

$$\textit{area of the elementary sector}=\frac{xdy-ydx}{2}.$$

On the determination of the expression for the radius of curvature in polar curves.

186. We have given, art. 149, the expression for the radius of curvature, referred to rectangular coordinates; which, assuming the positive sign for γ, is

$$\gamma=\frac{\left(1+\frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}\,.\,.\,.\,.\,.\,.\,.\,.\,.\,(107).$$

That this value of γ may be expressed in terms of the polar coordinates, we must eliminate the differential coefficients which enter into the formula (107) by means of the following equations,

$$x=u\cos\theta,\quad y=u\sin\theta;$$

which, being differentiated, and the results divided the one by the other, we shall obtain

$$\frac{dy}{dx}=\frac{du\sin\theta+u\cos\theta\,d\theta}{du\cos\theta-u\sin\theta\,d\theta};$$

and, representing the two terms of this fraction by m and n, we shall have

$$\left.\begin{aligned} m&=du\sin\theta+u\cos\theta\,d\theta,\\ n&=du\cos\theta-u\sin\theta\,d\theta.\end{aligned}\right\}\,.\,.\,.\,.\,.\,.\,(108);$$

and consequently

$$\frac{dy}{dx}=\frac{m}{n}\,.\,.\,.\,.\,.\,(109),$$

$$\frac{dy^2}{dx^2}=\frac{m^2}{n^2};$$

by means of which last equation we find for the numerator of the value of γ,

$$\left(1+\frac{dy^2}{dx^2}\right)^{\frac{3}{2}}=\left(\frac{m^2+n^2}{n^2}\right)^{\frac{3}{2}};$$

and raising each term of this fraction to the power $\frac{3}{2}$, and observing that the power $\frac{3}{2}$ of n^2 is n^3, we have

$$\left(1+\frac{d^2y}{dx^2}\right)^{\frac{3}{2}}=\frac{(m^2+n^2)^{\frac{3}{2}}}{n^3}\ \ldots\ldots (110).$$

Differentiating now the equation (109), we shall find

$$\frac{d^2y}{dx}=\frac{ndm-mdn}{n^2},$$

and dividing the first side of this equation by dx, and the second by n, which is equivalent to dx, we shall have

$$\frac{d^2y}{dx^2}=\frac{ndm-mdn}{n^3}\ \ldots\ldots (111).$$

By means of these values given by the equations (110) and (111), the equation (107) becomes

$$\gamma=\frac{(m^2+n^2)^{\frac{3}{2}}}{ndm-mdn}\ \ldots\ldots (112);$$

and we have now only to transform this equation into a function of θ and u; for which purpose we must determine first the value of n^2+m^2, by adding the squares of the equations (108), and reducing by means of the equation $\sin^2\theta+\cos^2\theta=1$; when we shall find

$$n^2+m^2=du^2+u^2d\theta^2\ \ldots\ldots (113).$$

To obtain the denominator of the equation (112), we must differentiate successively the equations (108), considering $d\theta$ as constant; and multiplying the results by n and m respectively, we shall find

$$ndm=nd^2u\sin\theta+2ndu\cos\theta d\theta-nu\sin\theta d\theta^2,$$
$$mdn=md^2u\cos\theta-2mdu\sin\theta d\theta-mu\cos\theta d\theta^2,$$

whence, subtracting the second equation from the first, we have

$$\left.\begin{aligned} ndm - mdn = {} & d^2u \ (n \sin\theta - m \cos\theta) \\ & + 2dud\theta \, (n \cos\theta + m \sin\theta) \\ & - ud\theta^2 \ (n \sin\theta - m \cos\theta) \end{aligned}\right\} \ldots \ (114).$$

Now, multiplying the first of the equations (107) by $\cos\theta$, and the second by $\sin\theta$, subtracting the one from the other, and reducing by means of the relation $\sin^2\theta + \cos^2\theta = 1$, we shall obtain

$$n \sin\theta - m \cos\theta = -ud\theta;$$

and the value of $n \cos\theta + m \cos\theta$ being determined in a similar manner, we shall find

$$n \cos\theta + m \cos\theta = du.$$

Substituting these values in equation (114), it becomes

$$udm - mdn = -ud^2u\,d\theta + 2du^2d\theta + u^2d\theta^3 \ \ldots \ (115),$$

and the values thus determined in equations (113) and (115) change the equation (112) into

$$\gamma = \frac{(du^2 + u^2d\theta^2)^{\frac{3}{2}}}{2du^2d\theta - ud^2ud\theta + u^2d\theta^3}.$$

On transcendental curves.

187. Curves are thus called which contain transcendental quantities or differential coefficients, and which, generally, cannot have their equations expressed in a finite number of algebraic terms. We will proceed to examine some of the most remarkable of these curves.

On the spiral of Archimedes or of Conon.

188. This curve is thus generated: whilst the radius AB (fig. 37) revolves about A as a centre, a point A, setting out from that centre, moves uniformly towards the extremity B of the radius, in such a manner, that when AB has completed a revolution about the centre A, the moveable point, which at the commencement of the rotation was at A, is arrived at B. The moveable point traces out in its course the spiral of Archimedes.

Let $AB = a$, arc $BN = \theta$, $AM = u$; we have then, from the preceding definition,

$$AM : AN :: \text{arc } NB : BCDB,$$

or

$$u : a :: \theta : 2\pi a,$$

whence we find

$$u = \frac{}{2\pi},$$

the equation to the curve, which, as we see, has not its co-ordinates rectangular.

When AB has made a complete revolution, the arc NB is equal to the circumference, and therefore $\theta = 2\pi a$, which changes the preceding equation into

$$u = \frac{2\pi a}{2\pi} = a.$$

If the point A continue to move on uniformly, the radius AB will make a second revolution around the centre A, and if we take $BB' = BA$, the moveable point, at the end of this second revolution, will be arrived at B′; when θ will be equal to $4\pi a$, and therefore $u = 2a$, and so on.

On the logarithmic spiral.

189. The logarithmic spiral is a polar curve, in which the angle AMT (fig. 81) formed by the radius vector with the tangent MT to the curve is constant; so that, representing the trigonometrical tangent of this angle by a, we have

$$\tan AMT = a.$$

But the triangle ATM, right-angled at A, gives us

$$1 : \text{tang } AMT :: AM : AT,$$

whence

$$\tan AMT = \frac{AT}{AM};$$

and replacing the radius vector AM by u and AT by the expression $\frac{u^2 d\theta}{du}$, which we found, art. 183, for the subtangent of a polar curve, we shall have

$$\tan \mathrm{AMT}, \text{ or } a = \frac{u\,d\theta}{du};$$

from which we deduce

$$\frac{a\,du}{u} = d\theta \;\ldots\ldots\; (116),$$

and integrating, we shall find

$$a \log u = \theta + constant.$$

Let e be the base of the Napierian system; supposing then that we have a system of tables such, that a may be the logarithm of e in that system, we may replace a by $\mathrm{L}e$, and $\mathrm{L}e \log u$ will be the logarithm of u in that system*; so that we shall have

$$\mathrm{L}u = \theta + constant.$$

190. The logarithmic spiral may be constructed by points in the following manner: having divided the circumference OO'O'' (fig. 83) into equal parts, draw radii to the points of division, and along these radii take the parts Am, Am', Am'', &c. in geometrical progression; then the points m, m', m'', &c. will belong to a logarithmic spiral. For if in the logarithmic spiral the parts mm', $m'm''$, $m''m'''$, &c. be taken exceedingly small, we may consider them as straight lines, and it will be easily seen that the triangles Amm', A$m'm''$, A$m''m'''$, &c. are similar, since the angles at A are equal by construction, and the angles mm'A, $m'm''$A, $m''m'''$A, &c. are so by the property of the curve; we have therefore from these triangles the series of proportions,

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* To demonstrate this, let e be the base of the Napierian system; we shall have then $u = e^{\log u}$; and taking the logarithms in the system of tables indicated by L,

$$\mathrm{L}u = \mathrm{L}\left(e^{\log u}\right) = \log u\,\mathrm{L}e.$$

$$\begin{array}{l} Am : Am' :: Am' : Am'', \\ Am' : Am'' :: Am'' : Am''', \\ \&c. : \&c. :: \&c. : \&c., \end{array}$$

which shows us that the ordinates Am, Am', Am'', &c. are in geometric progression.

191. In the logarithmic spiral, the normal is equal to the radius of curvature; for the radius of curvature, in a polar curve, having for its expression

$$\gamma = \frac{(du^2+u^2d\theta^2)^{\frac{3}{2}}}{2du^2d\theta - ud^2ud\theta + u^2d\theta^2},$$

if, in this formula, we substitute the values of du and d^2u, as derived from the equation (116) to the logarithmic spiral, which gives

$$du = \frac{ud\theta}{a},\ d^2u = \frac{du}{a}\,d\theta = \frac{ud\theta^2}{a^2}.$$

we shall have

$$\gamma = \frac{\left(\frac{u^2}{a^2}+u^2\right)^{\frac{3}{2}}}{\frac{u^2}{a^2}+u^2} = \sqrt{\frac{u^2}{a^2}+u^2}.$$

If now, in the expression for the normal, which is (art. 184)

$$\sqrt{u^2+\frac{du^2}{d\theta^2}},$$

we substitute the value of $\frac{du^2}{d\theta^2}$, we shall find the same value $\sqrt{\frac{u^2}{a^2}+u^2}$, which proves that in this curve the normal is equal to the radius of curvature; and since the two lines (art. 155) have the same direction, it follows that they must coincide.

192. This property will furnish us with the means of demonstrating that the evolute of the logarithmic spiral is another logarithmic spiral; for the point N of the normal being considered as belonging to the radius of curvature, and being taken at its extremity, will be in the evolute. Let u' and θ' be the coordinates of this point N (fig. 84); it will be easy to determine them in a function of the coordinates u and θ of the point M of the curve; for if oo' be an arc of a circle described with a radius *unity*, the abscissæ of the points M and N will differ from each other by that angle, which, since MAN is a right angle, will be equal to a quadrant of the circumference, and Fig. 84.

if, therefore, adopting the usual notation, we represent the quadrant of the circle described with radius unity by $\frac{\pi}{2}$, we shall have

$$\theta'=\theta+\frac{\pi}{2},$$

an equation, which, being differentiated, gives

$$d\theta'=d\theta.$$

For the polar ordinate u' of the point N of the evolute, since this ordinate is equal to the subnormal $\frac{du}{d\theta}$ of the logarithmic spiral, we must change $\frac{du}{d\theta}$ into u', in the equation of that curve, when we shall find $u=au'$, and, consequently, $du=adu'$; and substituting these values of $d\theta$, du, and u in the equation (116) of the logarithmic spiral, we shall find

$$a\frac{du'}{u'}=d\theta';$$

an equation which, being of the same form with the preceding one, shows us that the logarithmic spiral has for its evolute another logarithmic spiral.

On the hyperbolic spiral and the spirals comprised under the equation $u=a\theta^n$.

193. The property of the hyperbolic spiral is to have its subtangent constant; if therefore we represent this subtangent by a, and equate it to the expression for the subtangent (art. 183) of a polar curve, we shall have for the equation to the hyperbolic spiral

$$u^2\frac{d\theta}{du}=-a,$$

where a is taken negative, because we have then

$$-\frac{du}{u^2}=\frac{d\theta}{a};$$

an equation which, being integrated, gives

$$\frac{1}{u}=\frac{\theta}{a}+C,$$

and C, being an indeterminate quantity, may be replaced by a similar quantity $\frac{C'}{a}$, when we shall have

$$\frac{1}{u}=\frac{\theta}{a}+\frac{C'}{a},$$

and taking the origin of θ so that the abscissa $\theta+C'$ may be equal to some new abscissa θ, the equation will become

$$\frac{1}{u}=\frac{\theta}{a},$$

or

$$u=\frac{a}{\theta}\quad . \; . \; (117);$$

which shows that when $\theta=0$, $u=\infty$; and that consequently the radius vector which corresponds to the point where θ is 0, is an asymptote to the curve.

194. The equation (117) shows us, also, that the radius vector is in the inverse ratio of the abscissa, so that making successively $\theta=2\pi$, $\theta=4\pi$, $\theta=6\pi$, &c., we shall have for u the series of values $\frac{a}{2\pi}$, $\frac{a}{4\pi}$, $\frac{a}{6\pi}$, &c., which shows us that at the end of two revolutions the radius vector is reduced to the half of what it was at the end of the first, at the end of three revolutions to a third of that value, and so on.

195. The equations to the hyperbolic spiral and the spiral of Conon are particular cases of the equation $u=a\theta^n$; for by making $n=1$, and $a=\frac{1}{2\pi}$, we obtain the first, and by making $n=-1$, we obtain the second. Among the spirals determined by this equation we may notice the parabolic spiral, which is found by making $n=2$.

The logarithmic curve.

196. This is a curve in which the coordinates are rectan-

gular, and the abscissa is the logarithm of the ordinate; the equation to the curve is therefore

$$x = \log y,$$

whence we have

$$y = a^x,$$

and consequently

$$\frac{dy}{dx} = a^x \log a.$$

197. To discuss this equation; if we make $x = 0$, we shall find $y = 1$, and giving then to x values increasing and positive, y will go on continually increasing; but if we give to x a negative value, $-u$, we shall find $y = a^{-u} = \frac{1}{a^u}$, whence we see that, in respect of the negative abscissæ, y will be diminished the further we retire from the origin, but that the curve cannot reach the axis of x, produced in a negative direction, except at an infinite distance from the origin, in which case the equation $y = \frac{1}{a^u}$ will become $y = \frac{1}{a^\infty} = 0$; whence we may conclude that the axis of x, so produced, is an asymptote to the curve.

198. If, setting out from the origin, we take the equal ab-
Fig. 38. scissæ (fig. 38). $AP = u$, and $AP' = -u$, we shall find

$$PM = a^u,\ P'M' = a^{-u}, \text{and therefore } PM \times P'M' = 1.$$

199. The most remarkable property of this curve is, that its subtangent has a constant value; for the equation of the curve being differentiated, gives us $\frac{dy}{dx} = a^x \log a$, whence we find $\frac{a^x dx}{dy} = \frac{1}{\log a}$, or $\frac{y dx}{dy} = \frac{1}{\log a}$. But the first side of this equation expresses the subtangent of the curve, art. 69, and this subtangent therefore is constant.

On the cycloid.

200. The cycloid is the curve traced out by the motion of a point M (fig. 39), situated in the circumference of a circle which rolls along a straight line RC. In this motion from R to C, it is evident that all the points of the arc RM must come successively in contact with the straight line RA, until at length the point M itself comes in contact at A; the arc RM, consequently, will be equal to the straight line RA; and since also every point through which M passes is, by hypothesis, a point in the cycloid, A must be a point in that curve. Fig. 39.

Taking A, therefore, for the origin of the abscissæ, letting fall the perpendicular ME on the diameter BR, and making $AP=x$, $PM=y$, $BR=2a$, arc $MR=z$, $ME=u$, we shall have

$$AP=AR-PR,$$

or

$$x=\text{arc }MR-ME,$$

or

$$x=z-u \;.\;.\;.\; (118).$$

Proceeding first to eliminate the arc z, we must differentiate the preceding equation, which will give us

$$dx=dz-du \;.\;.\;.\; (119),$$

and to obtain the value of dz in a function of u, we must observe that between u and z we have the equation

$$u=\sin z,$$

which being differentiated, art. 42, we find

$$du=dz\cdot\frac{\cos z}{a},$$

whence we have

$$dz=\frac{adu}{\cos z},$$

and replacing, in this equation, the value of cos z given by the equation

$$\sin^2 z+\cos^2 z=a^2,$$

or

$$u^2+\cos^2 z=a^2,$$

we obtain

$$dz=\frac{adu}{\sqrt{a^2-u^2}},$$

which being substituted in the equation (119), it becomes

$$dx=\frac{adu}{\sqrt{a^2-u^2}}-du\ \ldots\ (120),$$

and we have now only to express u in a function of y. For this purpose, let O be the centre of the generating circle BMR (fig. 39), we have then

$$\mathrm{OE}=\sqrt{\mathrm{MO}^2-\mathrm{ME}^2},$$

or

$$a-y=\sqrt{a^2-u^2}\ \ldots\ (121),$$

which equation being squared and reduced, we deduce from it

$$u=\sqrt{2ay-y^2}\ \ldots\ (122),$$

and, by differentiating,

$$du=\frac{(a-y)\,dy}{\sqrt{2ay-y^2}}\ \ldots\ (123).$$

The equations (121) and (123) transform the equation (120) into

$$dx=\frac{ady}{\sqrt{2ay-y^2}}-\frac{(a-y)\,dy}{\sqrt{2ay-y^2}},$$

and reducing we find,

$$dx=\frac{ydy}{\sqrt{2ay-y^2}},$$

which is the equation of the cycloid.

201. The equation of the cycloid may be obtained also in a function of the arc, in the following manner: the equation $u=\sin z$ gives

$$z=\sin^{-1}.u,$$

so that putting for u its value derived from the equation (122), we have

$$z=\sin^{-1}.\sqrt{2ay-y^2},$$

and this value and that of u being substituted in the equation (118), it becomes

$$x=\sin^{-1}.\sqrt{2ay-y^2}-\sqrt{2ay-y^2}\,^{*}\ \ldots\ (124).$$

202. To discuss this equation, we will prove, first, that y cannot be either negative or greater than $2a$. For, in the first place, if we make $y=-y'$, the expression $\sin^{-1}.\sqrt{2ay-y^2}$ becomes $\sin^{-1}.\sqrt{-2ay'-y'^2}$, an imaginary value; and, secondly, if we make $y=2a+\delta$, the expression $\sin^{-1}\sqrt{2ay-y^2}$ becomes $\sin^{-1}.\sqrt{-2a\delta-\delta^2}$, which is also imaginary; and if, therefore, at a distance $EF=2a$, along the axis of x, we draw (fig. Fig. 40. 40) AB parallel to CD, the curve will be comprised within the parallels AB and CD.

The greatest value that y can have is $2a$; for if the generating circle be made to roll from A to C (fig. 41), the point

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[*] The sine here corresponds to a radius a; that of the tables, having unity for radius, would be

$$\frac{\sqrt{2ay-y^2}}{a};$$

and if, therefore, we wish to introduce this sine, we must write

$$x=a\sin^{-1}.\frac{\sqrt{2ay-y^2}}{a}-\sqrt{2ay-y^2}.$$

M, which was at first at A, will rise continually until it reaches B at the extremity of the diameter BD; when the abscissa AD will be equal to DEB, i. e. to the semi-circumference of the generating circle.

This result agrees with what is given us by the equation (124); for if we make $y=2a$, we find $x=\sin^{-1}.0$; but the arc whose sine is 0 must be one of the following; 0, DEB, 2DEB, 3DEB, &c., and we see that, in the present case, the arc is DEB.

If the point M, after having arrived at B, and so described the arc AB of the cycloid, continues to move on, it will describe a second arc BC, similar to the former; and if the generating circle be supposed to roll continually along the axis of the abscissæ, the point M will trace out an indefinite series of arcs of the cycloid, CB′C″ C″B″C‴, &c. (fig. 42). The generating circle may be supposed also to move from A towards A″, and we shall then have another indefinite series of arcs AB′A′, A′B″A″, &c.

It is the assemblage of all these arcs, which, in the most general sense of the word, constitutes the cycloid.

203. The normal at the point M, whose coordinates are x and y (fig. 43), is determined, art. 70, by the formula, Fig. 43.

$$\text{the } normal = y\sqrt{\frac{dy^2}{dx^2}+1}\,;$$

in which, if we substitute the value of $\frac{dy}{dx}$ derived from the equation to the cycloid, we shall find

$$\text{the } normal = y\sqrt{\frac{2ay-y^2}{y^2}+1} = \sqrt{2ay}.$$

To construct this value, if we draw the chord MD, fig. 43, we shall have

$$\text{DE} : \text{MD} :: \text{MD} : \text{DB},$$

or

$$y : \text{MD} :: \text{MD} : 2a,$$

whence consequently

$$\text{the cord } MD = \sqrt{2ay};$$

and since, by the property of the circle, BMD is a right angle, the chord MB will be perpendicular to the extremity of the normal MD, and therefore the chord MB produced will be a tangent at the point M in the cycloid; for we know that the tangent and the normal form a right angle with each other.

We may therefore construct the tangent at any point M by describing the generating semicircle BMD, and producing the chord BM; but instead of having to construct this circle at every point of the curve, it will be sufficient to describe the semicircle on the maximum ordinate BD of the cycloid; and having drawn through the given point M, the perpendicular ME on BD, to draw the chord BC; then the parallel MT to that chord will be the tangent required; as follows immediately from what has preceded.

204. To obtain the expression for the radius of curvature of the cycloid, we must deduce, from the equation of the curve, the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$, and substitute them in the expression for the radius of curvature, art. 150,

$$\gamma = -\frac{\left(1+\frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}.$$

to which we prefix the negative sign, because we know that the curve is concave to the axis of x.

Now the equation of the cycloid gives us at once

$$\frac{dy}{dx} = \frac{\sqrt{2ay-y^2}}{y} \quad \ldots . (125);$$

to obtain $\frac{d^2y}{dx^2}$, making $\frac{dy}{dx} = p$, we shall have

$$p=\frac{\sqrt{2ay-y^2}}{y}=\sqrt{\frac{2a}{y}-1},$$

and, by differentiating, art. 21, we shall find

$$dp=-\frac{\frac{2a}{y^2}dy}{2\sqrt{\frac{2a}{y}-1}}=-\frac{ady}{y\sqrt{2ay-y^2}},$$

$$\text{whence }\frac{dp}{dy}=-\frac{a}{y\sqrt{2ay-y^2}},$$

and multiplying this equation by equation (125), we shall obtain, art. 26,

$$\frac{dp}{dx}=-\frac{a}{y^2},\text{ or }\frac{d^2y}{dx^2}=-\frac{a}{y^2}.$$

By means of these values we have for the radius of curvature

$$\gamma=\frac{\left(\frac{2a}{y}\right)^{\frac{3}{2}}}{\frac{a}{y^2}}=\frac{(2a)^{\frac{3}{2}}}{\frac{ay^{\frac{3}{2}}}{y^2}}=\frac{2^{\frac{3}{2}}a^{\frac{1}{2}}}{y^{-\frac{1}{2}}},$$

and bringing y into the numerator,

$$\gamma=2^{\frac{3}{2}}a^{\frac{1}{2}}y^{\frac{1}{2}}=2.2^{\frac{1}{2}}a^{\frac{1}{2}}y^{\frac{1}{2}}=2\sqrt{2ay},$$

Fig. 45. and therefore the radius of curvature MM' (fig. 45) of the cycloid is double of the normal MR.

205. We shall obtain the equation of the evolute by substituting the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the formulæ, (art. 149)

$$y-\beta=-\frac{1+\frac{dy^2}{dx^2}}{\frac{d^2y}{dx^2}},$$

$$x-\alpha=\frac{\left(1+\frac{dy^2}{dx^2}\right)\frac{dy}{dx}}{\frac{d^2y}{dx^2}}=-(y-\beta)\frac{dy}{dx},$$

when we shall find

$$y-\beta=\frac{\frac{2a}{y}}{\frac{a}{y^2}}=2y,\ x-\alpha=-2\sqrt{2ay-y^2};$$

and therefore

$$\beta=-y,\ \alpha=x+2\sqrt{2ay-y^2},$$

or (fig. 46)

$$QM'=MP,\ \alpha=AP+2ME.$$

Observing that AP+ME=AR=arcMR, the last of these equations may be written thus,

$$\alpha=\text{arc}MR+ME\ \ldots.\ (126);$$

and producing BR, taking RL=BR=$2a$, and describing on RL the semi-circumference, RM′L, that semi-circumference will pass through the point M′, on account of the equal chords M′R and MR, and we shall have

$$\text{arc}\,MR=\text{arc}M'R \text{ and } ME=M'E,$$

which values being substituted in the equation (126), we shall find

$$\alpha=\text{arc}M'R+M'E',$$

and therefore

$$\alpha=\text{arc}M'R+\sqrt{2\alpha\beta-\beta^2}\ \ldots.\ (127),$$

the equation which exists between AQ=α, and QM′=β, the coordinates of any point M′ in the evolute.

If now we produce the ordinate CD=$2a$ to a point A′ (fig. 46), so that DA′ shall be also equal to $2a$, and through Fig. 46.
the point A′ draw the parallel A′D′ to AD, and transfer the

origin from A to A', making $A'Q'=\alpha'$, $Q'M'=\beta'$, we shall have for the abscissa

$$A'Q'=AD-AQ,$$

or

$$\alpha'=\tfrac{1}{2}\text{ generating circumference}-AQ,$$

or

$$\alpha'=\pi a-\alpha;$$

in regard to the ordinate β, we have

$$M'Q'=A'D-QM',$$

or

$$\beta'=2a-\beta;$$

from these equations, therefore, we derive

$$\alpha=\pi a-\alpha',\ \beta=2a-\beta',$$

and substituting these values in equation (127), it becomes

$$\pi a-\alpha'=\text{arc}\,M'R+\sqrt{2a\beta'-\beta'^2};$$

or

$$\pi a-\alpha'=\text{arc}\,RM'L-\text{arc}\,M'L+\sqrt{2\alpha\beta-\beta'}$$
$$=\pi a-\text{arc}\,M'L+\sqrt{2a\beta'-\beta'^2},$$

and, consequently,

$$\alpha'=\text{arc}\,M'L-\sqrt{2a\beta'-\beta'^2},$$

which is the equation to a cycloid, and therefore the evolute of a cycloid is another cycloid.

206. It may be demonstrated by synthesis also, that the evolute AA' (fig. 46) is a cycloid; for we have Fig. 46.

$$\text{arc}\,LM'+\text{arc}\,RM'=\pi\alpha,$$

and therefore

$$\text{arc}\,LM'=\pi a-\text{arc}\,RM';$$

on the other hand,

$$\text{arc}\,RM=\text{arc}\,MR$$
$$=AR,\ (\text{art. }199),$$

which value being substituted in the preceding equation, we shall have

$$\text{arc LM}' = \pi a - \text{AR} = \text{AD} - \text{AR},$$

or

$$\text{arc LM}' = \text{LA}',$$

which is a property of the cycloid.

On the change of the independent variable.

207. When an equation is given containing differential coefficients, we can eliminate them only by means of the equation of the curve to which we wish to apply the formula; thus when we have the formula

$$\frac{\left(1+\frac{dy^2}{dx^2}\right)}{\frac{d^2y}{dx^2}} \cdot \frac{dy}{dx},$$

and it is asked what it becomes when the curve is a parabola, we deduce from the equation $y=ax^2$ of the parabola, the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$, and substitute them in the formula, when the differential coefficients will disappear. If $\frac{dy}{dx}$ and $\frac{d^2y}{dx}$ be considered as two unknown quantities, we must have in general two equations in order to eliminate them from any proposed formula, and these equations will be obtained by differentiating the equation of the curve twice successively.

208. When, by the operations of Algebra, dx has been removed from under dy, as in the following formula,

$$\frac{y(dx^2+dy^2)}{dx^2+dy^2-yd^2y} \ldots . (128),$$

the substitution is made by considering dx, dy, and d^2y, as the unknown quantities; and since to eliminate them we must have in general an equal number of equations, it does not appear at first sight that this elimination can be effected; for the differentiation of the equation of the curve can furnish us with only two equations betwixt dx, dy, and d^2y; but we must observe that when, by means of these two equations, we have eliminated dy and d^2y, dx^2 will be found as a common factor in the formula, and will consequently disappear. If, for example, the curve be a parabola represented by $y=ax^2$, this equation being differentiated twice successively, we shall obtain,

$$dy = 2ax\,dx,\ d^2y = 2a\,dx^2,$$

and these values being substituted in the formula (128), and the common factor dx^2 suppressed, there results,

$$\frac{y(1+4a^2x^2)}{4a^2x^2-2ay}.$$

209. The reason why dx^2 thus becomes a common factor will be easily seen; for when in a formula containing originally $\frac{d^2y}{dx^2}$ and $\frac{dy}{dx}$, the denominator of $\frac{d^2y}{dx^2}$ is made to disappear, all the terms, except those in $\frac{d^2y}{dx^2}$ and $\frac{dy}{dx}$, must acquire the common factor dx^2, the terms affected by $\frac{d^2y}{dx^2}$ will no longer contain dx, whilst the terms affected by $\frac{dy}{dx}$ will involve the first power of dx, for the product of $\frac{dy}{dx}$ by dx^2 is $dydx$. When, therefore, we differentiate the equation of the curve, and so obtain results of the form $dy = \mathrm{M}dx$, $d^2y = \mathrm{N}dx^2$, these values, being substituted in the terms involving d^2y and $dydx$, will change them, like the other terms, into products of dx^2.

210. What we have said of a formula containing differentials of the two first orders will apply to those in which the differentials are of higher orders, and it follows, therefore, that by differentiating the equation of the curve as often as it shall be necessary, we may always clear a proposed formula of the differentials contained in it.

211. This will not be the case, if, besides the differentials which we have been considering, the formula should contain terms in d^2x, in d^3x, &c.; for, suppose that there entered into the formula the following differentials dx, dy, d^2y, d^2x, and that by differentiating the equation represented by $y = fx$ twice in succession, we had deduced from it the equations

$$\mathrm{F}(x,\ y,\ dy,\ dx) = 0,\ \mathrm{F}(x,\ y,\ dx,\ dy,\ d^2x,\ d^2y) = 0,$$

with these two equations then we could eliminate only two of the three differentials dy, d^2x, d^2y, and we see that it would be impossible to make all the differentials disappear from the formula. In this case, therefore, there is a tacit condition expressed by the differential d^2x, which is, that the variable x is itself to be considered as a function of a third variable which does not appear in the formula, and which is called the *independent variable;* this will become evident, if we observe that the equation $y = fx$ may be derived from the system of the two equations,

$$x = Ft,\ y = \varphi t,$$

betwixt which t has been eliminated; thus the equation $y = a \cdot \frac{(x-c)^2}{b^2}$ reverts to the system of the two equations

$$x = bt + c,\ y = at^2,$$

and we may conceive that x and y ought to vary by virtue of the increment which t receives; but this hypothesis, that x and y vary according to the increment given to t, supposes that there are certain relations betwixt x and t, and y and t, one of which relations is arbitrary; for the equation, which we represent in general by $y = fx$, being, for example, $y = a\frac{(x-c)^2}{b^2}$, if we establish betwixt x and t the arbitrary relation $x = \frac{t^3}{c^2}$, this value being substituted in the equation $y = a\frac{(x-c)^2}{b^2}$, will change it into $y = a\frac{(t^3-c^3)^2}{b^2c^4}$, an equation which, combined with the one $x = \frac{t^3}{c^2}$ must, by elimination, give $y = a \cdot \frac{(x-c)^2}{b^2}$, the only condition to which we need have regard in the choice of the variable t.

212. We may, therefore, determine the independent variable t arbitrarily. For instance, the chord, the arc, the abscissa, or the ordinate, may be taken for that independent variable; and if t represent the arc of the curve, we must have $dt = \sqrt{dx^2 + dy^2}$; if t represent the chord, and the origin be at the vertex of the curve, we shall have $t = \sqrt{x^2 + y^2}$; lastly, t may be the abscissa or the ordinate, and we shall have then $t = x$, or $t = y$.

213. The choice of one or other of these hypotheses is indispensable, in order that a formula containing differentials may be freed from them; and if it is not always made, it is because we tacitly suppose that the independent variable has been determined. For instance, in the most ordinary case in which the formula contains only the differentials dx, dy, d^2y, d^3y, &c., the hypothesis is, that the abscissa has been taken for the independent variable t, for then there results

$$t = x,\ \frac{dx}{dt} = 1,\ \frac{d^2x}{dt^2} = 0,\ \frac{d^3x}{dt^3} = 0,\ \&c.$$

and we see that the formula cannot contain any differentials of x of a higher order than the first.

214. To establish the formula in all its generality, it is necessary, then,

from what has preceded, that x and y should be functions of a third independent variable t; and that we should have, (art. 26),

$$\frac{dy}{dt}=\frac{dy}{dx}\frac{dx}{dt};$$

from which equation we deduce

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}} \ldots\ldots (129);$$

and taking the second differential of y, and applying to the second side of this equation the rule for fractions, art. 16, we shall find

$$\frac{d^2y}{dx}=\frac{\frac{dx}{dt}\frac{d^2y}{dt}-\frac{dy}{dt}\frac{d^2x}{dt}}{\frac{dx^2}{dt^2}}.$$

In this expression dt has two uses; one to indicate what the independent variable t is, and the other to enter as an algebraic symbol. If we keep in mind that t is the independent variable, dt may be considered only in its second character; and then suppressing dt^2 as a common factor, the preceding expression will be reduced to

$$\frac{d^2y}{dx}=\frac{dx\,d^2y-dy\,d^2x}{dx^2};$$

and, by dividing by dx, will become

$$\frac{d^2y}{dx^2}=\frac{dx\,d^2y-dy\,d^2x}{dx^3}.$$

215. Proceeding in the same manner with the equation (129), we see that by taking t for the independent variable, the second side of the equation becomes identical with the first; and, consequently, when we take t for the independent variable, the only change we have to make in the formula containing the differential coefficients $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$, is that of replacing the second differential coefficient by $\frac{dx d^2y-dy\,d^2x}{dx^3}$. To apply these considerations to the radius of curvature, which, art. 186, is given by the equation

$$\gamma=\frac{\left(1+\frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{\frac{d^2y}{dx^2}},$$

if we wish to have the value γ, in the case in which t is the independent variable, we must change this equation into

$$\gamma=\frac{\left(1+\frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{\frac{dx\,d^2y-dy\,d^2x}{dx^3}};$$

and observing that the numerator reduces itself to $\frac{(dx^2+dy^2)^{\frac{3}{2}}}{dx^3}$ we shall have

$$\gamma=\frac{(dx^2+dy^2)^{\frac{3}{2}}}{dxd^2y-dyd^2x}\,\cdots\cdots\cdots\,(130).$$

216. This value of γ supposes then that x and y are functions of a third independent variable; if x should be this variable, i. e. if $t=x$, we should have $d^2x=0$, and the expression (130) would be reduced to its ordinary form,

$$\gamma=\frac{(dx^2+dy^2)^{\frac{3}{2}}}{dxd^2y}=\frac{\left(1+\frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}.$$

217. But if, instead of taking x for the independent variable, we wished that y should be that variable, this condition would be expressed by the equation $y=t$, and by differentiating this equation twice successively, we should have

$$\frac{dy}{dt}=1,\quad \frac{d^2y}{dt^2}=0.$$

The first of these equations tells us only that y is the independent variable, and makes no change in the formula; but the second shows us that d^2y ought to be 0, and then the equation (130) is reduced to

$$\gamma=-\frac{(dx^2+dy^2)^{\frac{3}{2}}}{dyd^2x}.$$

218. We may observe, that when x is the independent variable, we have $d^2x=0$, an equation which shows us that dx is constant; whence it follows that, generally, the variable which is considered as independent, has always a constant differential.

219. If, lastly, we take the arc for the independent variable, we shall have the equation

$$dt=\sqrt{dx^2+dy^2},$$

and squaring and dividing by dt^2, this will give us

$$\frac{dx^2}{dt^2}+\frac{dy^2}{dt^2}=1\,;$$

differentiating this equation, we must consider, art. 218, dt as constant, since t is the independent variable; and applying the usual rules, we shall find

$$\frac{2dxd^2x}{dt^2}+\frac{2dyd^2y}{dt^2}=0\,;$$

whence we deduce

$$dxd^2x=-dyd^2y\,;$$

and, consequently, if we substitute the value of d^2x, or that of d^2y, in the equation (130), we shall have, in the first case,

$$\gamma=\frac{(dx^2+dy^2)^{\frac{3}{2}}}{(dx^2+dy^2)d^2y}dx=\frac{\sqrt{dx^2+dy^2}}{d^2y}dx\,;$$

in the second case,

$$\gamma=-\frac{(dx^2+dy^2)^{\frac{3}{2}}}{(dx^2+dy^2)d^2x}dy=-\frac{\sqrt{dx^2+dy^2}}{d^2x}dy.$$

220. In what has preceded, we have considered only the two differential coefficients $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$; but if the formula should contain the differential coefficients of higher orders, we must, by methods analogous to those just employed, determine the values of $\frac{d^3y}{dx^3}$, $\frac{d^4y}{dx^4}$, &c., which belong to the case in which x and y are functions of some third independent variable.

On the method of infinitesimals.

221. The ideas which we have of infinity reduce themselves to this proposition: *A quantity is not infinite so long as it admits of augmentation.* If, consequently, we have a quantity $x+a$, and x become infinite, a must be suppressed, otherwise it would be supposed that x might be increased by the quantity a, which is contrary to our definition.

222. This being a fundamental proposition, it has been endeavoured to demonstrate it in a more satisfactory manner, as follows:

Let the equation be

$$\frac{1}{a}+\frac{1}{x}=\mathrm{M} \; \ldots\ldots \; (131);$$

which, being multiplied by the product ax, becomes

$$x+a=\mathrm{M}ax \; \ldots\ldots \; (132).$$

Supposing, now, that x becomes infinite, the fraction $\frac{1}{x}$, having reached its last degree of diminution, is evidently reduced to zero, and therefore the equation (131) becomes

$$\mathrm{M}=\frac{1}{a};$$

and this value being substituted in equation (132), we obtain

$$x+a=x;$$

which shows, that when x is infinite, $x+a$ is reduced to x.

223. The quantity a, in regard to which x is infinite, is what we term an *infinitesimal,* in respect of x.

224. Since we consider here only the ratio of quantities, the preceding demonstration holds good when x has a finite value, provided only that a be infinitely small in respect of x. The theory of fractions will serve to render the truth of this still more evident; for if we compare the finite quantity b with the fraction $\frac{b}{z}$, it is clear that the more the denominator z be increased, the more the fraction is diminished; so that when z becomes infinite the fraction will become absolutely 0, and, as such, ought to be suppressed before b, which will be then infinite in respect of $\frac{b}{z}$.

225. Although two quantities be themselves infinitely small, it does not follow that their ratio should be 0; for

$$\frac{a}{\infty} : \frac{b}{\infty} :: a : b.$$

It will be seen likewise that two infinitely small quantities may measure each other, as well as two exceedingly large quantities; whence, representing the two infinitely small quantities by dx and dy, it follows that their ratio $\frac{dy}{dx}$ will not necessarily be 0; a result agreeable to what we have obtained by the consideration of limits.

226. When a quantity x is infinitely small, in respect of a finite magnitude a, the square, x^2, is infinitely small in respect of x; for the proportion

$$1 : x :: x : x^2,$$

shows us that x^2 is contained in x as often as x is contained in unity, i. e. an infinite number of times. It may be demonstrated, in like manner, by means of the proportion

$$x : x^2 :: x^2 : x^3;$$

that x^2 being infinitely small in respect of x, the term x^3 must be infinitely small in respect of x^2; and for this reason it is that infinitesimals have been divided into different orders; thus, in the preceding examples, x is an infinitesimal of the first order, x^2 one of the second order, x^3 one of the third order, and so on.

227. We may observe, that if x be infinitely small in respect of a, it will still be so when multiplied by a finite quantity b; for, since x may be considered as a fraction of which the denominator is infinite, we may represent x by $\frac{c}{\infty}$; and whether we have $\frac{c}{\infty}$ or $\frac{bc}{\infty}$, these quantities will be no less 0 in respect of a.

228. In the same manner that an infinitesimal of the first order ought to be suppressed when placed by the side of a finite quantity, which it cannot augment, we must leave out an infinitesimal of the second order, which appears by the side of an infinitesimal of the first order; and so on.

If, for example, we have the expression

$$a + by + cy^2 + dy^3,$$

and y be an infinitesimal of the first order, and consequently cy^2 be one of the second, and dy^3 one of the third order; we must first leave out dy^3, because it cannot augment cy^2; and since cy^2 cannot augment by, it must be left out in its turn; lastly, we must leave out by also, since this infinitesimal of the first order cannot augment the finite quantity a, and there will remain a.

229. Two infinitesimals, x and y, give for their product an infinitesimal of the second order; for from the product xy is derived the proportion

$$1 : y : . \ x : xy;$$

which shows us, that since y is an infinitesimal in respect to 1, xy will be an infinitesimal in respect to x, i. e. will be an infinitesimal of the second order.

230. It might be proved, in like manner, that the product of three infinitesimals of the first order gives an infinitesimal of the third order.

231. We may now explain the theory of differentiation according to the method of infinitesimals. For this purpose, if we suppose that in a function of x the variable x receives an increment infinitely small, so that x becomes $x+dx$, the difference between this new value and the former will be the differential of the function.

232. For example, to find the differential of ax, this function becoming $a(x+dx)=ax+adx$, if we subtract from it ax, there will remain adx for the differential.

233. Let it be proposed also to find the differential of ax^3; we must then subtract ax^3 from $a(x+dx)^3$, and developing and reducing, we shall find $3ax^2dx+3axdx^2+adx^3$. Now adx^3 being an infinitesimal of the third order, cannot augment $3axdx^2$, and consequently we must leave out the term adx^3; in like manner, $3axdx^2$, which is an infinitesimal of the second

order, must be suppressed, because $3ax^2dx$ is an infinitesimal of the first order; and there will remain $3ax^2dx$ for the differential.

234. On the same principle we might differentiate every other function of x, taking care to suppress the infinitesimals of higher orders, which comes to the same thing with retaining only the first term of the development, as was done in the method of limits. For example, to find the differential of fx, instead of writing

$$\frac{f(x+h)-fx}{h}=\mathrm{A}+\mathrm{B}h+\mathrm{C}h^2+\&c.$$

which, in the case of the limit, gives $\frac{dfx}{dx}dx=\mathrm{A}dx$ for the differential, we should have

$$f(x+dx)=fx+\mathrm{A}dx+\mathrm{B}dx^2+\mathrm{C}dx^3+\ \&c.;$$

subtracting the primitive function, there would remain

$$\mathrm{A}dx+\mathrm{B}dx^2+\mathrm{C}dx^3+\ \&c.;$$

and since infinitesimals of the higher orders are to be suppressed, we should retain only the term $\mathrm{A}dx$, which would be the differential sought.

235. To find the differential of the product of two variables, y, z, we will suppose that when x becomes $x+dx$, y becomes $y+dy$, and z becomes $z+dz$. The product yz will then be changed into $(y+dy)\ (z+dz)$, and developing and subtracting yz, there will remain $ydz+zdy+dydz$; in which result the last term, being an infinitesimal of the second order, must be suppressed, and we shall have, for the differential of yz, the expression $ydz+zdy$.

236. From this last differential we may deduce that of the product of any greater number of variables, and then that of x^m, by the same processes that we adopted when employing the method of limits.

237. The differential of a^x will also be obtained very easily,

when we have the development of a^{x+dx}, and this development will be found as that of a^{x+h} was, art. 36; we must then determine the value of $a^{x+dx}-a^x$, and retaining only the first term, reject all the others as being infinitesimals of higher orders than the term which we keep. From the differential of a^x, we shall deduce, as we did before, that of log x.

238. In regard to the differential of sin x, we have sin $(x+dx)-\sin x=\sin x.\cos dx+\sin dx.\cos x-\sin x$, and the arc dx being infinitely small,

$$\cos dx=1, \text{ and } \sin dx=dx;$$

from which values we find

$$d.\sin x=\cos x.dx.$$

239. The problem of tangents may almost be said to have given birth to the differential calculus. We will show how this problem is resolved by the method of infinitesimals.

Fig. 47.

Let PM and P′M′ (fig. 47) be two ordinates indefinitely near to each other, and MO a parallel to the axis of x; then the tangent MT may be considered as the prolongation of the element MM′ of the curve, since that element, being exceedingly small, may be supposed to be a straight line. Call AP, x; PM, y; then the increment of x will be PP′$=dx$, that of y will be M′O$=dy$; and the indefinitely small triangle MM′O being similar to the triangle MPT, we have

$$\text{M}'\text{O} : \text{MO} :: \text{MP} : \text{PT},$$

or

$$dy : dx :: y : \text{PT};$$

and, therefore,

$$\text{PT}=y\frac{dx}{dy}.$$

We shall find then the normal, the tangent, and the equations to those lines, just as in art. 70 and 71.

240. To obtain the differential of an arc, we must consider the arc included between the coordinates PM and P′M′, taken

indefinitely near to each other, as a straight line; and then, calling the whole arc s, MM′ will be ds, and the right-angled triangle MM′O will give

$$\mathrm{MM}^2=\mathrm{MO}^2+\mathrm{M'O}^2,$$

or

$$ds^2=dx^2+dy^2;$$

and taking the square root,

$$ds=\sqrt{dx^2+dy^2}.$$

241. The differential of the arc of a curve, whose coordinates are polar, is also found very readily by the consideration of infinitesimals. For let RR′ and MN (fig. 82) be two circular arcs described, one with radius *unity*, and the other with radius u, and subtending the indefinitely small angle M′AM, formed by two radii vectores; then the triangle NM′M may be considered as rectilinear and right-angled at M; and we shall have therefore

$$\mathrm{MM'}=\sqrt{\mathrm{NM'}^2+\mathrm{NM}^2};$$

and observing that M′N $=du$, and that MN is equal to $ud\theta$, from the proportion

$$1 : d\theta :: u : \mathrm{MN},$$

we may replace NM′ and MN by their values; when, putting ds in place of MM′, we shall have

$$ds=\sqrt{du^2+u^2d\theta^2}.$$

The same triangle MM′N, compared with the triangle M′AT, will give us the value of the subtangent in a polar curve from the proportion

$$\mathrm{M'N} : \mathrm{MN} :: \mathrm{AM'} : \mathrm{AT},$$

or replacing AM′ by AM, from which it differs only by an infinitesimal,

$$du : ud\theta :: u : \mathrm{AT};$$

whence we derive

$$\mathrm{AT}=\frac{u^2d\theta}{du}.$$

On the method of Lagrange, for demonstrating the principles of the Differential Calculus, without the consideration of limits, infinitesimals, or any evanescent quantity.

242. We have seen of what utility Taylor's theorem was when it was wished to develop functions in form of a series.

Lagrange, observing the great facility with which the principles of differentiation might be deduced from this theorem (*note second*), arrived at its demonstration without making use of the Differential Calculus, by a process which we shall proceed to modify in the manner following:

Let $y = f(x+h)$; from the nature of this function, when h is made $=0$, $f(x+h)$ must necessarily be reduced to fx; and this will be the case if the part containing h in this equation be a multiple of h. Let this part be represented by P, we shall have then

$$f(x+h) = fx + Ph;$$

and since P also may be a function of h, if we represent by p what P becomes when $h=0$, and by Qh the part depending on h, we shall have also $P = p + Qh$; and continuing this reasoning we shall have the series of equations

$$\begin{aligned} y &= fx + Ph, \\ P &= p + Qh, \\ Q &= q + Rh, \\ \&c. &= \&c. + \&c. \end{aligned}$$

Putting now the value of P, given by the second equation, in the first there will result

$$y = fx + ph + Qh^2;$$

putting in this result the value of Q, given by the third equation, we shall have

$$y = fx + ph + qh^2 + Rh^3;$$

and continuing this process, and putting $f(x+h)$ in place of y, we shall have, generally,

$$f(x+h) = fx + ph + qh^2 + rh^3 + sh^4 + \&c. \ldots (133).$$

243. The expression $f(x+h)$ represents, generally, the function which is not yet reduced to the form of a series; if in this function we change x into $x+i$, we shall have the same result as if we had changed h into $h+i$; for since this func-

tion cannot contain x without that variable being followed immediately by h, a term, such as $A(x+h)^m$, for instance, when we change x into $x+i$, will become $A(x+h+i)^m$, a quantity the same with $A(x+h+i)^m$, which would result from the substitution of $h+i$ in place of h in the function $A(x+h)^m$. What we have said of this term will apply to all the others; and it follows, therefore, that, on the two hypotheses, the first side of the equation (133) must give identical results, and that consequently the development $fx+ph+qh^2+\&c.$, must give the same result whether we replace x by $x+i$, or h by $h+i$.

244. Substituting, first, $h+i$ for h in $fx+ph+qh^2+\&c.$, we shall have

$$fx+p(h+i)+q(h+i)^2+r(h+i)^3+\&c. \;\ldots\; (134);$$

and taking only the two first terms of each of these binomials, there will result

$$fx+ph+pi+qh^2+2qhi+rh^3+3rh^2i+\&c. \;\ldots\; (135).$$

To obtain now the result of the substitution of $x+i$ for x, in the expression $fx+ph+qh^2+rh^3+\&c.$, we must observe, that since h always shows itself wherever it exists in this series, it cannot enter into fx, and the coefficients p, q, r, &c., and that these quantities therefore can only contain x, and must be considered as its functions; and since the equation (133) holds good for every function of x, the substitution $x+i$ for x will change

$$\begin{array}{rllllll}
fx \text{ into } fx & +pi & +qi^2 & +ri^3 & +si^4 & +\&c., \\
p \text{ into } p & +p'i & +p''i^2 & +p'''i^3 & +p^{iv}i^4 & +\&c., \\
q \text{ into } q & +q'i & +q''i^2 & +q'''i^3 & +q^{iv}i^4 & +\&c., \\
r \text{ into } r & +r'i & +r''i^2 & +r'''i^3 & +r^{iv}i^4 & +\&c., \\
s \text{ into } s & +s'i & +s''i^2 & +s'''i^3 & +s^{iv}i^4 & +\&c., \\
\&c. & \&c. & \&c. & \&c. & \&c. &
\end{array}$$

where it is almost needless to observe that by the accented letters we represent the coefficients of the different powers of i in these developments. Substituting these values of fx, p, q, r, s,

&c. in the series $fx+ph+qh^2+$&c., we shall obtain

$$fx+pi+qi^2+ri^3+\&c.+(p+p'i+p''i^2+\&c.)h$$
$$+(q+q'i+q''i^2+\&c.)h^2+(r+r'i+r''i^2+\&c.)h^3+\&c.\ldots(136).$$

245. Since this development must be identical (art. 243) with that denoted by (135), the terms which contain the same powers of h in these developments must be equal (*note third*); and consequently, if we compare the terms involving hi, h^2i, h^3i, &c., we shall find

$$p'=2q,\ q'=3r,\ r'=4s,\ \&c.\ \ldots\ldots\ (137).$$

246. We have seen, art. 244, that p was generally a function of x; representing therefore p by $f'x$ and the coefficient of h in the development of $f'(x+h)$ by $f''x$; and representing in like manner the coefficient of h in the development of $f''(x+h)$ by $f'''x$, and so on, we shall have the equations

$$\left.\begin{array}{l} f(x+h)=fx\ \ +hf'x\ \ +\text{terms in } h^2,\ h^3,\ \&c.\\ f'(x+h)=f'x\ +hf''x+\text{terms in } h^2,\ h^3,\ \&c.\\ f''(x+h)=f''x+hf'''x+\text{terms in } h^2,\ h^3,\ \&c.\\ \quad\&c. = \quad\ \&c. \qquad \&c. \qquad\quad \&c. \end{array}\right\}\ldots(138).$$

247. Now, by hypothesis, we have, art. 246, $p=f'x$; if, therefore, in this equation, we make $x=x+h$, we shall have

$$p+p'h+p''h^2+p'''h^3+\&c.=f'(x+h)\ \ldots\ldots\ (139);$$

and putting in this equation the value of $f'(x+h)$, given by the second of the equations (138), we shall obtain

$$p+p'h+p''h^2+\&c.=f'x+hf''x+\text{terms in } h^2,\ h^3,\ \&c.;$$

which equation being true, whatever h be, the terms involving the same powers of h must be equal, and consequently

$$p'=f''x.$$

This value of p' will change the first of the equations (137) into $f''x=2q$, whence we shall derive

$$q=\tfrac{1}{2}f''x;$$

and if in this equation we change x into $(x+h)$, there will result

$$q+q'h+q''h^2+\&c=\tfrac{1}{2}f''(x+h);$$

whence, putting for $f''(x+h)$ its development, given by the third of the equations (138), we shall have

$$q+q'h+q''h^2+\&c.=\tfrac{1}{2}(f''x+hf'''x+\text{terms in } h^2,\ h^3,\ \&c.);$$

and comparing the terms which involve the first power of k, we shall find $q'=\frac{1}{2}f'''x$, a value which, being substituted in the second of the equations (137), will change it into $\frac{1}{2}f'''x=3r$, whence we derive

$$r=\tfrac{1}{3}\cdot\tfrac{1}{2}\cdot f'''x.$$

Proceeding thus, we shall find successively all the other coefficients of the equation (133); and substituting in that equation the values of p, q, r, &c., we shall have

$$f(x+h)=fx+hf'x+\frac{h^2}{1.2}f''x+\frac{h^3}{2.3}f'''x+\&c.\ldots.\ (140).$$

248. If now we consider the first of the equations (138), we shall see that $f'x$, being the coefficient of h in the development of $f(x+h)$, is what we have designated by $\frac{d.fx}{dx}$, or by $\frac{dy}{dx}$; observing, in the same manner, the second of the equations (138), we perceive that the coefficient $f''x$ of the first power of h in the development of $f'(x+h)$ ought to be represented by $\frac{d.f'x}{dx}$, i. e. by $\frac{d.\frac{dy}{dx}}{dx}=\frac{d^2y}{dx^2}$, and so on; when, consequently, putting these values of fx, $f'x$, $f''x$, &c., in the equation (140), we shall find

$$f(x+h)=fx+\frac{dy}{dx}h+\frac{d^2y}{dx^2}\frac{h^2}{1.2}+\frac{d^3y}{dx^3}\frac{h^3}{2.3}+\&c.\ldots..\ (141).$$

249. It is thus that we arrive at Taylor's theorem, without making use of the differential Calculus. The expression $\frac{dy}{dx}$, which enters into this formula, is the symbol of the operation by which we obtain the coefficient of h in the development of $f(x+h)$; and, after this coefficient is found, the expressions $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$, &c., indicate to us that the same process repeated will make known to us the coefficients of the other powers of h; so that we require only to know by the rules of algebra, what $\frac{dy}{dx}$ ought to be for each function. If, for example, it were asked what $\frac{dy}{dx}$ is for the function x^m, we should develop $(x+h)^m$ by the binomial theorem, which would give $x^m+mx^{m-1}h+$ &c.; and since $\frac{dy}{dx}$ must indicate the coefficient of the first power of h in this development, we should have $\frac{dy}{dx}=mx^{m-1}$. Thus the whole is reduced to the being able to find, by analytical processes, the development of the different sorts of functions which algebra can present; and these processes will not differ in any way from those which we have given for developing the different functions which, by their combination, give all the others. It is thus that we have given the developments of a^{x+h}, of log $(x+h)$, and of cos $(x+h)$, &c.

250. Here, then, is a third method, by which the principles of the differential calculus appear demonstrated in a manner independent of every consideration of limits, infinitesimals, or evanescent quantities; but, nevertheless, this method does not exclude that of limits, for when we come to its applications, and wish, for instance, to determine the volumes, the surfaces, or the lengths of curves, or to obtain the expressions for the

sub-tangents, sub-normals, &c., we are constantly obliged to have recourse to limits or infinitesimals.

251. In considering the developments of the different functions $(x+h)^m$, a^{x+h}, $\log(x+h)$, $\sin(x+h)$, &c., which algebra presents to us; since these functions are exceedingly limited in number, it is easy to perceive that, in their developments, the coefficient of the first power of h is not either 0 or infinity, so long at least as x retains its indeterminate value; and this moreover results generally from the preceding demonstration. For suppose that we had $p=0$ in the equation

$$f(x+h)=fx+ph+qh^2+rh^3+\&c.;$$

there would then be two cases; the value of x, which p contains, would either be given by an identical equation, or by one that was not so; in the latter case $p=0$ would represent an equation of a certain degree, and this equation would give a limited number of values of x, which would be contrary to our hypothesis, which admits for x any value whatever; but if $p=0$, i. e. if $f'x=0$, be an equation identical in respect of x *, making $x=x+h$, we should have still $f'(x+h)=0$; and since h would enter wherever x does, this equation, considered in respect of h, would still be identically 0, or, in other words, this equation would hold good whatever were the value of h; and it would therefore be the same with its development, which, according to the equation (139), is

$$p+p'h+p''h^2+p'''h^3+\&c.=0:$$

but when an equation of this sort is 0, independently of h, the coefficients of the different powers of h must be separately

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* The case in which p does not contain x is comprised in this; for if the value of p, which is 0, be represented by $a-a$, we may consider it as . . . . $(a-x)-(a-x)$.

0 (*note third*), and we must consequently have

$$p'=0,\ p''=0,\ p'''=0,\ \&c.$$

Substituting these values in the equations

$$p'=2q,\ p''=3r,\ p'''=4s,\ \&c.$$

which result from the identity of the terms affected by the same powers of ih, i^2h, i^3h, &c., in the series 134 and 136, we should obtain

$$q=0,\ r=0,\ s=0,\ \&c.;$$

and, since also $p=0$, the equation (133) would be reduced to

$$f(x+h)=fx,$$

and it would be necessary, therefore, that $x+h$, put in place of x, should produce no change in the function, which would require the function to be identical or constant; for we know that if fx were, for instance, of the form x^2-x^2, or of the form $c+x^2-x^2$, the substitution of $x+h$ in place of x would give always the same result, and we see that, in the first case, the function would be identical, and in the second, would be reduced to a constant. From this it follows, that the coefficient of the first power of h cannot be 0 in the general development of $f(x+h)$.

It would be no less absurd to suppose that coefficient infinite, for the second side of the equation (133) becoming infinite, the first side would be so likewise, i. e. we should have $f(x+h)=\infty$; and since $f(x+h)$ is composed of $(x+h)$ as fx is of x, the term which in $f(x+h)$ renders that expression infinite, must also render fx infinite. For example, if $f(x+h)$ contained a term such as $\frac{A}{(x+h)-(x+h)}$, which is infinite, it is evident that we must have in fx the term $\frac{A}{x-x}$, which would be also infinite. It follows, therefore, that the proposed function would be infinite, which we do not at all suppose.

252. The expressions $f'x$, $f''x$, $f'''x$, &c., are what Lagrange calls the *prime function*, the *second function*, the *third function*, &c. of fx, and generally they are the derived functions. Lagrange indicates also the derived functions in another manner, by replacing $\frac{dy}{dx}$ by y', $\frac{d^2y}{dx^2}$ by y'', $\frac{d^3y}{dx^3}$ by y''', and so on.

On the case in which Taylor's theorem fails.

253. In general, when in a function of x we put $x+h$ in place of x, the form of the function remains the same, since $x+h$ enters wherever x did; thus, when fx contains a radical, $f(x+h)$ will contain it also; if, for example, we have

$$fx=bx^2+\frac{a}{\sqrt{x}},$$

the same radical will be found in the expression

$$f(x+h)=b(x+h)^2+\frac{a}{\sqrt{x+h}}.$$

254. This would not always be the case, if we should give a particular value to x; for instance, if $\sqrt[3]{x-a}$ should enter into fx, $f(x+h)$ would necessarily contain the term

$$\sqrt[3]{x+h-a};$$

and the hypothesis of $x=a$ would cause the term $\sqrt[3]{x-a}$, which appears in fx, to vanish, whilst the same hypothesis would reduce $\sqrt[3]{x+h-a}$, which enters into $f(x+h)$, to $\sqrt[3]{h}=h^{\frac{1}{3}}$; so that the development of $f(x+h)$ would then contain a surd quantity, which does not exist in fx, and consequently could not be developed according to integral powers of h.

This impossibility would manifest itself by the infinite values assumed by the differential coefficients; for example, if we had the equation

$$y=\sqrt[3]{x-a},$$

we should find, by differentiating,

$$\frac{dy}{dx}=\frac{1}{3}(x-a)^{-\frac{2}{3}}=\frac{1}{3\sqrt[3]{(x-a)^2}},$$

and we see that the value of the differential coefficient becomes infinite, when we make $x=a$.

255. Let, generally,

$$f(a+h)=\mathrm{A}+\mathrm{B}h+\mathrm{C}h^2+\mathrm{D}h^3+\ldots.\mathrm{M}h_n+\mathrm{N}h^{n+\frac{1}{z}}+\text{\&c.}\ldots(142)$$

be the development which we are supposed to obtain by making $x=a$, and in which $n+\frac{1}{z}$ represents a quantity lying between n and $n+1$; we will proceed to show that the differential coefficient of the order $n+1$ is infinite. For this purpose, considering a as a variable, we know, art. 53 and 54, that we have

$$\frac{df(a+h)}{da}=\frac{df(a+h)}{dh},\ \frac{d^2f(a+h)}{da^2}=\frac{d^2f(a+h)}{dh^2}\ \text{\&c.}\ .\ (143);$$

now, by differentiating the equation (142) successively in respect of h, and, for the sake of brevity, representing by M′, N′; M″, N″, &c. what the coefficients M and N then become, we shall have

$$\frac{df(a+h)}{dh}=\mathrm{B}+2\mathrm{C}h+3\mathrm{D}h^2\ldots+\mathrm{M}'h^{n-1}+\mathrm{N}'h^{n+\frac{1}{z}-1}+\text{\&c.},$$

$$\frac{d^2f(a+h)}{dh^2}=2\mathrm{C}+2.3\,\mathrm{D}h\ldots.+\mathrm{M}''h^{n-2}+\mathrm{N}''h^{n+\frac{1}{z}-2}+\text{\&c.},$$

&c. &c. &c.;

replacing the first sides of these last equations by their values given by the equation (143), we shall obtain

$$\frac{df(a+h)}{da}=\mathrm{B}+2\mathrm{C}h+3\mathrm{D}h^2\ldots+\mathrm{M}'h^{n-1}+\mathrm{N}'h^{n+\frac{1}{z}-1}+\ldots(144),$$

$$\frac{d^2f(a+h)}{da^2}=2\mathrm{C}+2.3.\mathrm{D}h\ldots\ldots+\mathrm{M}''h^{n-2}+\mathrm{N}''h^{n+\frac{1}{z}-2}+\ldots(145);$$

and making $h=0$ in the equations (142), (144), and (145), &c., we have

$$fa = A,\ \frac{dfa}{da} = B,\ \frac{d^2fa}{da^2} = 2C,\ \&c.,$$

which will suffice for determining the coefficients A, B, C, &c. of the equation (142).

This being premised, from the inspection of the equations (144) and (145), we see that, since every differentiation diminishes n by unity, when we have arrived at the n^{th} differentiation, we must have

$$\frac{d^nf(a+h)}{da^n} = \ldots Ph^{n-n} + Qh^{n-n+\frac{1}{z}} + \&c. = P + Qh^{\frac{1}{z}} + \&c.$$

and from the next differentiation, we shall find

$$\frac{d^{n+1}f(a+h)}{da^{n+1}} = Rh^{\frac{1}{z}-1} + \&c.;$$

but $\frac{1}{z}$ being less than unity, $\frac{1}{z}-1$ is a negative number, and the preceding equation may therefore be written thus,

$$\frac{d^{n+1}f(a+h)}{da^{n+1}} + \frac{R + \&c.}{h^{1-\frac{1}{z}}},$$

consequently, when we put $h=0$ to determine the coefficient of one of the terms of the equation (142), we shall find

$$\frac{d^{n+1}fa}{da^{n+1}} = \frac{R}{0} = \infty;$$

and it will be the same when we wish to determine the differential coefficients of a higher order.

It follows from this theorem, that *when we make* x=a *in the development of* f(x+h), *if there be a fractional power of* h *in the development, and it lie betwixt the terms affected by* h^n *and* h^{n+1}, *we shall not be able to determine the terms of the series of Taylor beyond that of the order* n; *all the other terms will become infinite.*

256. A function of x represented by fx being given, if we wish to determine the development of $f(x+h)$, on the hypothesis of $x=a$, we must, as we know, calculate the terms of the series

$$fx + \frac{dy}{dx}h + \frac{d^2y}{dx^2}\frac{h^2}{1.2} + \&c.;$$

but if in making this calculation we find that one of the differential coefficients becomes infinite on the hypothesis of $x=a$, we must no longer seek to obtain the development of $f(x+h)$ by Taylor's series, but employ the following method: We must put $x+h$ in place of x in the function fx, then the term which contains $x-a$ in the denominator, will contain $x-a+h$, and will no longer become infinite when we put $x=a$, but will produce a term affected with a fractional power of h.

257. For example, let the function be

$$fx=2ax-x^2+a\sqrt{x^2-a^2},$$

by differentiating, we find

$$\frac{dy}{dx}=2(a-x)+\frac{ax}{\sqrt{x^2-a^2}},$$

and substituting these values and those of $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$, &c., in Taylor's formula, art. 55, we obtain

$$f(x+h)=2ax-x^2+a\sqrt{x^2-a^2}+\left[2(a-x)+\frac{ax}{\sqrt{x^2-a^2}}\right]h+\text{ \&c.}$$

in which expression, when $x=a$; the term multiplied by h becomes infinite, and therefore the development is no longer possible.

In this case, according to the preceding rule, we must put $x+h$ for x in the equation

$$fx=2ax-x^2+a\sqrt{x^2-a^2},$$

and we shall find

$$f(x+h)=2ax+2ah-x^2-2xh-h^2+a\sqrt{x^2+2xh+h^2-a^2},$$

an equation which, on the hypothesis of $x=a$, becomes

$$f(a+h)=a^2-h^2+a\sqrt{2ah+h^2},$$

or

$$f(a+h)=a^2-h^2+a\sqrt{h}\sqrt{2a+h},$$

and developing by the binomial theorem, and, for brevity's sake, representing by A, B, C, &c., the coefficients given by that formula, we have

$$\sqrt{2a+h}=(2a+h)^{\frac{1}{2}}=A+Bh+Ch^2+Dh^3+\text{ \&c};$$

when, substituting, we find

$$f(a+h)=a^2-h^2+aA\sqrt{h}+aBh\sqrt{h}+aCh^2\sqrt{h}+\text{ \&c.}$$

We see, from this example, that by putting $x+h$ for x, in the function, and making $x=a$, we may introduce one or more fractional powers of h; we then develop, separately, the terms which are susceptible of it, either by the binomial theorem or otherwise, and substitute these terms in the value of $f(a+h)$, which then gives the development.

258. When x continues indeterminate, Lagrange has demonstrated that the development of $f(x+h)$ cannot contain terms with fractional powers of h. For suppose that we had

$$f(x+h)=fx+ph+qh^2+\ \ldots\ +K\sqrt[3]{h};$$

then, since $K\sqrt[3]{h}$ allows of three values, M, N, P, we shall have the three developments of $f(x+h)$:

$$\begin{aligned}f(x+h)&=fx+ph+qh^2+\ \ldots\ +M,\\ f(x+h)&=fx+ph+qh^2+\ \ldots\ +N,\\ f(x+h)&=fx+ph+qh^2+\ \ldots\ +P.\end{aligned}$$

But since fx must contain the same roots that $f(x+h)$, art. 253, does, fx must also have three different values Q, R, S, and substituting successively these values in place of fx, we shall find

$$\begin{aligned}f(x+h)&=Q+ph+qh^2+\ \ldots\ +M,\\ f(x+h)&=Q+ph+qh^2+\ \ldots\ +N,\\ f(x+h)&=Q+ph+qh^2+\ \ldots\ +P,\\ f(x+h)&=R+ph+qh^2+\ \ldots\ +M,\\ f(x+h)&=R+ph+qh^2+\ \ldots\ +N,\\ f(x+h)&=R+ph+qh^2+\ \ldots\ +P,\\ f(x+h)&=S+ph+qh^2+\ \ldots\ +M,\\ f(x+h)&=S+ph+qh^2+\ \ldots\ +N,\\ f(x+h)&=S+ph+qh^2+\ \ldots\ +P,\end{aligned}$$

so that the expression $f(x+h)$, when developed, will have nine different values, whilst, undeveloped, it can have only as many as fx contains, and consequently three on the present hypothesis; thus we cannot suppose that the development of $f(x+h)$ contains a fractional power of h, without falling into a contradiction.

259. It is easy also to demonstrate that $f(x+h)$ cannot contain in its development a term affected with a negative power of h, for if it should contain a term such as Mh^{-n}, we should have

$$f(x+h)=fx+ph+qh^2\ \ldots\ +\frac{M}{h^n},$$

and, by making $h=0$, the first side would become fx, whilst the second, instead of being reduced to fx also, would become infinite, on account of the term $\frac{M}{h^n}$, which it contains.

260. The same thing would happen, if the development should contain a term affected by the logarithm of h; for if we had, for example, a term such as $A \log h$, this term, when we made $h=0$, would become $A \log 0$; and since the logarithm of 0 is $-\infty$, the term $A \log h$ would then be infinite; whence fx would become so likewise, which is contrary to our hypothesis.

END OF THE DIFFERENTIAL CALCULUS.

ELEMENTS

OF THE

DIFFERENTIAL AND INTEGRAL CALCULUS.

INTEGRAL CALCULUS.

On the integration of monomial differentials.

261. THE object of the Integral Calculus is to find the function, which, being differentiated, will produce a given differential. To commence with the most simple case, we will proceed to integrate the expression $x^m dx$; with which view, differentiating the expression x^{m+1}, we shall find

$$d.x^{m+1}=(m+1)x^m dx,$$

whence we shall deduce

$$\frac{d.x^{m+1}}{m+1}=x^m dx;$$

and since the constant $m+1$ has no effect on the differentiation, the preceding equation may be written thus,

$$d.\frac{x^{m+1}}{m+1}=x^m dx;$$

consequently, the quantity which, when differentiated, will give $x^m dx$ is $\frac{x^{m+1}}{m+1}$. To indicate this operation, we put before the differential the characteristic $\int$, which signifies *sum* or *integral**, so that we may write

$$\int x^m dx=\frac{x^{m+1}}{m+1}.$$

262. Hence we deduce this general rule: *to integrate* x^{m}dx *we must increase the index by unity, and divide by the index so increased and by the differential.*

263. Take, for example, $\frac{adx}{x^3}$; we shall have then

$$\int\frac{adx}{x^3}=\int adx\,.\,x^{-3}=\frac{ax^{-3+1}}{-3+1}=\frac{ax^{-2}}{-2}=-\frac{a}{2x^2};$$

we shall find similarly that

$$\int dx\sqrt[3]{x^2}=\int x^{\frac{2}{3}}dx=\frac{x^{\frac{2}{3}+1}}{\frac{2}{3}+1}=\frac{x^{\frac{5}{3}}}{\frac{5}{3}}=\frac{3x^{\frac{5}{3}}}{5}.$$

264. We must observe, that when we differentiate $a+x^m$, we find $mx^{m-1}dx$, as though we had differentiated only x^m; consequently, in integrating, we must add a constant to the integral. Thus, in the preceding examples, we must write

$$\int\frac{adx}{x^3}=-\frac{a}{2x^2}+C,\ \int dx\sqrt[3]{x^2}=\frac{3}{5}x^{\frac{5}{3}}+C.$$

265. This constant C, which ought to vanish by the dif-

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* This term *sum*, for designating the integral, has been introduced by the ancient geometricians, because, according to the method of infinitesimals, they considered a function y as the sum of all the infinitesimal increments.

Fig. 48. For example, we see that the ordinate being MP (fig. 48), we have

$$MP=ab+a'b'+a''b''+a'''b'''+a^{IV}M;$$

that is to say, y is equal to the sum of all the infinitesimal increments, represented each by dy.

ferentiation, is in general arbitrary, unless from the nature of the problem it be determinate.

If, for example, we have the equation $y=ax^2-b$, which is that of a parabola CBD (fig. 49), whose origin is in A, and Fig. 49. we deduce from it

$$dy=2ax\,dx,$$

we shall find, by integrating,

$$y=ax^2+C\;\ldots\ldots(1),$$

and this equation may belong to an infinite number of parabolas, according to the value of C.

But if we wished that of all these parabolas CBD, C′B′D′, C″B″D″, &c., the curve which has for its equation $y=ax^2+C$, should be a parabola passing through the point E, whose coordinates are

$$y=0,\quad x=\mathrm{AE}=\sqrt{\frac{b}{a}},$$

it is necessary that, on making $x=\sqrt{\frac{b}{a}}$, we should have $y=0$, which will reduce the equation (1) to

$$0=b+C,$$

whence we shall deduce

$$C=-b,$$

and substituting this value in equation (1), we shall obtain

$$y=ax^2-b,$$

as we had before the differentiation.

266. When the nature of the problem does not determine the constant, we may dispose of the constant as in the following example; having found that the integral of $x^m dx$ is

$$y=\frac{x^{m+1}}{m+1}+C,\;\ldots\ldots(2),$$

we will give to x a determinate value b, when the second side of our equation will become

$$\frac{b^{m+1}}{m+1}+C \dots (3),$$

and C being arbitrary, we may determine this constant on the condition that the expression (3) is 0, or, which is the same thing, on the condition that we have $y=0$ when $x=b$; then the equation (2) will become

$$0=\frac{b^{m+1}}{m+1}+C,$$

whence we shall deduce the value of C, which being substituted in equation (2) will give us

$$y=\frac{x^{m+1}-b^{m+1}}{m+1} \dots . (4).$$

267. There is only one case to which the rule of art. 262 for integrating $x^m dx$ will not apply; it is that in which $m=-1$, for then the formula (4) becomes

$$y=\frac{x^0-b^0}{0}=\frac{0}{0};$$

in this case, therefore, we should have to make use of the rule of art. 81 for determining the true value of the integral; but we may avoid this inconvenience by observing that . . . . $x^{-1}dx=\frac{dx}{x}$, and that this expression $\frac{dx}{x}$ is the differential of $\log x$, consequently, we shall have

$$\int\frac{dx}{x}=\log x+C.$$

Certain complex differentials whose integration may be effected by the rule of the art. 262.

268. We have seen, art. 22, that the differential of a polynomial is formed of the sum of the differentials of its terms;

reciprocally the integral of a polynomial will be equal to the sum of the integrals of the terms which compose it.

For example,

$$\int\left(adx-\frac{bdx}{x^3}+xdx\sqrt{x}\right)=\int adx-\int\frac{bdx}{x^3}+\int x^{\frac{3}{2}}dx+C,$$

or, by performing the requisite operations (art. 262),

$$\int\left(adx-\frac{bdx}{x^3}+xdx\sqrt{x}\right)=ax+\frac{b}{2x^2}+\frac{2}{5}x^{\frac{5}{2}}+C.$$

We have added but one constant, because each term giving a constant to the integration, we may represent the sum of these constants by a single letter.

269. Every polynomial, such as $(a+bx+cx^2+\&c.)^n dx$, may be integrated by the same rule, when n is a positive integer; since we have only to raise the polynomial to the power indicated by n, and integrate each term separately.

For example, to integrate $(a+bx)^2dx$, we shall have

$$\int(a+bx)^2dx=\int(a^2+2abx+b^2x^2)dx$$

$$=a^2x+abx^2+\frac{b^2x^3}{3}+C.$$

270. When we have an expression such as $(Fx)^n dFx$, composed of two factors, one of which is the differential of the part Fx within the vinculum, we must put $Fx=z$, and consequently $dFx=dz$; when, substituting, we shall find

$$(Fx)^n dFx=z^n dz,$$

and integrating,

$$\int(Fx)^n dFx=\frac{z^{n+1}}{n+1}+C=\frac{(Fx)^{n+1}}{n+1}+C.$$

To give an example, let the expression be

$$(a+bx+cx^2)^{\frac{2}{3}}(bdx+2cxdx);$$

since $bdx+2cxdx$ is the differential of the quantity contained within the vinculum, putting

$$a+bx+cx^2=z,$$

we shall have, by differentiating,

$$bdx+2cxdx=dz,$$

and therefore

$$(a+bx+cx^2)^{\frac{2}{3}}(bdx+2cxdx)=z^{\frac{2}{3}}dz,$$

whence the integral of the expression will be

$$\frac{3}{5}z^{\frac{5}{3}}+\mathrm{C}=\frac{3}{5}(a+bx+cx^2)^{\frac{5}{3}}+\mathrm{C}.$$

271. If one of the factors be the differential of the other, except as to the constants, we may still integrate by the same process. Let the expression, for instance, be

$$(a+bx^2)^{\frac{1}{2}}mxdx \quad . \; . \; . \quad (5);$$

since we see that the differential of $a+bx^2$, which is $2bxdx$, differs from $mxdx$ only by the constant, we will put . . . . . $a+bx^2=z$, and consequently $2bxdx=dz$, whence we shall deduce $xdx=\frac{dz}{2b}$, and substituting these values in the expression (5), we shall obtain

$$(a+bx^2)^{\frac{1}{2}}mxdx=\frac{m}{2b}z^{\frac{1}{2}}dz\,;$$

and, by integrating, there will result

$$\int(a+bx^2)^{\frac{1}{2}}mxdx=\frac{m}{3b}z^{\frac{3}{2}}+\mathrm{C}=\frac{m}{3b}(a+bx^2)^{\frac{3}{2}}+\mathrm{C}.$$

272. The same transformation will also apply for reducing certain differentials to logarithmic forms; if we had, for ex-

ample, $\frac{adx}{a+bx}$, making $a+bx=z$, we should deduce $dx=\frac{dz}{b}$; substituting, we should have

$$\int\frac{adx}{a+bx}=\int\frac{a}{b}\frac{dz}{z}=\frac{a}{b}\int\frac{dz}{z}=\frac{a}{b}\log z+\mathrm{C},$$

and putting for z its value,

$$\int\frac{adx}{a+bx}=\frac{a}{b}\log(a+bx)+\mathrm{C}.$$

Proceeding in the same manner for $\frac{adx}{a-bx}$, we shall find that the integral of this expression is

$$\int\frac{adx}{a-bx}=-\frac{a}{b}\log(a-bx)+\mathrm{C}.$$

Integration by circular arcs.

273. Let the arc CB (fig. 50) $=z$, and its sine CE $=x$; we have then $x=\sin z$, and differentiating, there will result Fig. 50.

$$dx=\cos z dz,$$

whence we shall get

$$dz=\frac{dx}{\cos z};$$

but the equation $\cos^2 z+\sin^2 z=1$ gives us

$$\cos z=\sqrt{1-\sin^2 z}=\sqrt{1-x^2};$$

substituting this value, therefore, in that of dz, we shall obtain

$$dz=\frac{dx}{\sqrt{1-x^2}},$$

and consequently we shall find, by integrating,

$$\int\frac{dx}{\sqrt{1-x^2}}=z+\mathrm{C}\ \ldots\ldots\ (6).$$

To determine the constant, suppose that when $x=0$, we have

$$\int\frac{dx}{\sqrt{1-x^2}}=0\,;$$

since, then, according to fig. 50, the arc z, represented by CB, is 0 at the same time with the sine x, the equation (6) will, on this hypothesis, be reduced to $0=C$, and consequently

$$\int\frac{dx}{\sqrt{1-x^2}}=\sin^{-1}x.$$

274. On the above integral may be made to depend that of

$$\frac{dx}{\sqrt{a^2-x^2}}\,;$$

for by dividing the two terms of the fraction by a, we shall have

$$\int\frac{\frac{dx}{a}}{\sqrt{1-\frac{x^2}{a^2}}}=\int\frac{d.\frac{x}{a}}{\sqrt{1-\frac{x^2}{a^2}}}\,;$$

and since this integral is composed of $\frac{x}{a}$, in the same manner that $\int\frac{dx}{\sqrt{1-x^2}}$, is of x, it follows that

$$\int\frac{dx}{\sqrt{a^2-x^2}}=\sin^{-1}\frac{x}{a}.$$

275. In the second place, let z be the arc CD (fig. 50) whose cosine AG is x; we shall have then

$$x=\cos z\,;$$

and, by differentiating,

$$dx=-\sin z dz\,;$$

whence we shall derive

$$dz = -\frac{dx}{\sin z};$$

and putting for sin z its value drawn from the equation

$$\cos^2 z + \sin^2 z = 1,$$

we shall obtain

$$dz = -\frac{dx}{\sqrt{1-\cos^2 z}};$$

or, since $\cos z = x$,

$$dz = -\frac{dx}{\sqrt{1-x^2}};$$

when, integrating, we shall have

$$\int \frac{-dx}{\sqrt{1-x^2}} = \cos^{-1}x = \text{arc DC} + \text{C} \ldots . (7).$$

To determine the constant, suppose that when $x = 0, \int -\frac{dx}{\sqrt{1-x^2}}$ is also 0; in this case, then, equation (7) is reduced to

$$0 = \cos^{-1}0 + \text{C} \ldots .. (8);$$

but in order that the cosine AG of the arc CD may be 0, that arc must become

$$\text{DB} = \tfrac{1}{4}\,\textit{circumference} = \frac{\pi}{2};$$

putting, therefore, $\frac{1}{2}\pi$ in place of $\cos^{-1}0$ in equation (8), we have

$$0 = \tfrac{1}{2}\pi + \text{C},$$

which gives us

$$\text{C} = -\tfrac{1}{2}\pi;$$

and substituting this value in equation (7), we obtain

$$\int -\frac{dx}{\sqrt{1-x^2}} = -(\tfrac{1}{2}\pi - \text{arc DC}) = -\text{arc CB}.$$

276. We have seen, art. 45, that

$$d.\tan x = \frac{dx}{\cos^2 x};$$

if, therefore, we make $x = \tan z$, we shall find

$$dx = \frac{dz}{\cos^2 z};$$

and consequently

$$dz = dx \,.\, \cos^2 z:$$

but the proportion

$$\cos z : 1 :: 1 : \text{secant } z,$$

giving

$$\cos z = \frac{1}{\sec z},$$

we shall have

$$\cos^2 z = \frac{1}{\sec^2 z} = \frac{1}{1 + \tan^2 z} = \frac{1}{1 + x^2};$$

whence, substituting this value, we shall find

$$dz = dx \,.\, \frac{1}{1 + x^2};$$

and, integrating, we shall have

$$\int \frac{dx}{1 + x^2} = z + C.$$

Taking then the integral on the supposition that the integral vanishes when $x = 0$, z becomes 0, and we have

$$0 = C;$$

and therefore

$$\int \frac{dx}{1 + x^2} = \textit{arc whose tangent is } x.$$

277. We may bring under this form the integral

$$\int \frac{dx}{a^2 + x^2};$$

for, dividing the two terms of the fraction by a^2, we may write it thus,

$$\int\frac{\frac{dx}{a^2}}{1+\frac{x^2}{a^2}}=\frac{1}{a}\int\frac{\frac{dx}{a}}{1+\frac{x^2}{a^2}};$$

and since this integral is composed of $\frac{x}{a}$, as $\frac{1}{a}\int\frac{dx}{1+x^2}$ is of x, we shall have

$$\int\frac{dx}{a^2+x^2}=\frac{1}{a}\tan^{-1}\frac{x}{a}.$$

Let also x be the versine DG: then the cosine and versine being together equal to unity, we shall have

$$x+\cos z=1;$$

and, by differentiating,

$$dx=dz\,.\sin z\,,$$

whence we deduce

$$dz=\frac{dx}{\sin z}:$$

but

$$\sin z=\sqrt{1-\cos^2 z}=\sqrt{(1-\cos z)(1+\cos z)}=\sqrt{x(2-x)}\,;$$

substituting, therefore, we have

$$dz=\frac{dx}{\sqrt{2x-x^2}};$$

and, by integrating,

$$\int\frac{dx}{\sqrt{2x-x^2}}=\text{versin}^{-1}x.$$

We add no constant, because, supposing that the integral vanishes when x is 0, z is also 0.

278. When we wish to have the value of the integral for a

particular value of x, we must proceed as in the following example.

Suppose that the integral of $\frac{dx}{1+x^2}$ is required, when $x=7$; the radius therefore being 1, the tangent will be 7; and since the tables of sines are constructed with a radius = ten thousand millions, the tangent relatively to that radius will be ten thousand million times greater, and consequently the value of that tangent will be 7×10000000000. The logarithm of the tabular tangent will therefore be expressed by

$$\begin{aligned}\log 10000000000 + \log 7 &= 10 + \log 7,\\ &= 10 + 0{,}845098,\\ &= 10{,}845098;\end{aligned}$$

and looking out for this logarithm in the tables of sines, we shall see that it corresponds to an arc of

$$90^\circ\, 96, \textit{ of the decimal scale,}$$

or,

$$81^\circ\, 52', \textit{ of the sexagesimal scale.}$$

To find the numerical value of this arc, on the supposition of *radius* = 1, we must observe that, on this hypothesis, the circumference = 6,283 . . . . ; and, consequently, we shall have

$$400^\circ : 90^\circ 96' :: 6{,}283 \ldots : \textit{arc sought} = 1{,}42 \ldots$$

or

$$360^\circ : 81^\circ 52' :: 6{,}283 \ldots : \textit{arc sought} = 1{,}42 \ldots$$

Integration by parts.

279. In taking the differential of a product of two variables, by the process of art. 14, we find

$$d.uv = udv + vdu;$$

whence, integrating and transposing, there results

$$\int udv = uv - \int vdu;$$

and it is to this formula that we refer the differentials which we wish to integrate by the method of parts.

280. If, for example, we were unacquainted with the integral of $x^m dx$, we should put $x^m = u$, $dx = dv$, and we should have

$$\int x^m dx = x^{m+1} - \int x . m x^{m-1} dx = x^{m+1} - m \int x^m dx \, ;$$

whence, collecting the terms affected by $x^m dx$, we have

$$(m+1) \int x^m dx = x^{m+1},$$

and therefore

$$\int x^m dx = \frac{x^{m+1}}{m+1} + C.$$

281. Let the integral be

$$\int dx \log x \, ;$$

make

$$\log x = u, \;\; dx = dv,$$

and we have

$$\int dx \log x = x \log x - \int dx = x \log x - x + C = (\log x - 1) x + C.$$

282. As a last example, let it be proposed to integrate

$$dx \sqrt{a^2 - x^2},$$

making

$$\sqrt{a^2 - x^2} = u, \text{ and } dx = dv,$$

we shall find first

$$\int dx \sqrt{a^2 - x^2} = x \sqrt{a^2 - x^2} + \int \frac{x^2 dx}{\sqrt{a^2 - x^2}} \;\; \ldots . \; (9) \, ;$$

and we must now look out for another value of

$$\int dx \sqrt{a^2 - x^2} \, ;$$

for which purpose, by multiplying this last expression by

$$\frac{\sqrt{a^2 - x^2}}{\sqrt{a^2 - x^2}},$$

we shall have the identical equation

$$\int dx\sqrt{a^2-x^2}=\int\frac{a^2dx}{\sqrt{a^2-x^2}}-\int\frac{x^2dx}{\sqrt{a^2-x^2}};$$

whence, integrating the first of the expressions on the second side, we shall obtain

$$\int dx\sqrt{a^2-x^2}=a^2\sin^{-1}\frac{x}{a}-\int\frac{x^2dx}{\sqrt{a^2-x^2}},$$

and adding this equation to equation (9), we shall find

$$2\int dx\sqrt{a^2-x^2}=x\sqrt{a^2-x^2}+a^2\sin^{-1}\frac{x}{a};$$

and therefore

$$\int dx\sqrt{a^2-x^2}=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+C.$$

We see, from these examples, that when, generally, we have an expression such as $\int vdu$, the method of parts renders the proposed integral dependent on that of $\int udv$, and that, consequently, this method of integration is not always applicable.

Integration by series.

283. Let Xdx be a differential in which X represents a function of x; if we develop X in a series,

$$Ax^\alpha+Bx^\beta+Cx^\gamma+Dx^\delta+Ex^\epsilon+\&c.,$$

arranged according to the exponents α, β, γ, &c., we shall have

$$\int Xdx=\int(Ax^\alpha+Bx^\beta+Cx^\gamma+Dx^\delta+Ex^\epsilon+\&c.)dx$$

$$=\frac{Ax^{\alpha+1}}{\alpha+1}(^*)+\frac{Bx^{\beta+1}}{\beta+1}+\frac{Cx^{\gamma+1}}{\gamma+1}+\frac{Dx^{\delta+1}}{\delta+1}+\&c.+C.$$

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[*] Should one of the exponents α, β, γ, &c., be equal to -1, the term so affected must be integrated by logarithms.

284. Let us take, for example, the fraction $\frac{dx}{a+x}$, which is the differential of $\log(a+x)$: this fraction may be written thus:

$$\frac{1}{a+x}\times dx,$$

and we must now find the development of $\frac{1}{a+x}$, which might be done by means of division; but without performing this operation, we may deduce the development required from a formula easily remembered, which is this,

$$\frac{1}{1-z}=1+z+z^2+z^3+z^4+\&c. \ldots (10).^*$$

For the fraction $\frac{1}{a+x}$ may be put under the form

$$\frac{1}{a}\times\frac{1}{1+\frac{x}{a}},$$

when, to develop $\frac{1}{1+\frac{x}{a}}$, we have only to change z into $-\frac{x}{a}$, in the formula (10), and we shall have

$$\frac{1}{1+\frac{x}{a}}=1-\frac{x}{a}+\frac{x^2}{a^2}-\frac{x^3}{a^3}+\&c.;$$

and therefore

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* This formula having been found by division, it might be supposed that it would be better at once to divide 1 by $(a+x)$, but I have observed that when a particular formula is fixed in the memory, it is easier to deduce from it different developments, than to repeat the operation in each case.

$$\frac{1}{a}\times\frac{1}{1+\frac{x}{a}} \text{ or } \frac{1}{a+x}=\frac{1}{a}-\frac{x}{a^2}+\frac{x^2}{a^3}-\frac{x^3}{a^4}+\&\text{c.};$$

consequently,

$$\int\frac{dx}{a+x}=\int\left(\frac{1}{a}-\frac{x}{a^2}+\frac{x^2}{a^3}-\frac{x^3}{a^4}+\&\text{c.}\right)dx;$$

whence, by integrating each term separately, we shall obtain

$$\int\frac{dx}{a+x}=\frac{x}{a}-\frac{x^2}{2a^2}+\frac{x^3}{3a^3}-\&\text{c.}+\text{C}\ \ldots\ (11);$$

and observing that the first side of this equation is a logarithmic differential, art. 272, we shall have

$$\log\,(a+x)=\frac{x}{a}-\frac{x^2}{2a^2}+\frac{x^3}{3a^3}-\&\text{c.}+\text{C}\ \ldots\ (12).$$

To determine the constant, we must observe that when $x=0$, this equation is reduced to $\log a=0+\text{C}$; which value being substituted for C, the equation (12) will become

$$\log(a+x)=\log a+\frac{x}{a}-\frac{x^2}{2a^2}+\frac{x^3}{3a^3}-\&\text{c.}\ \ldots\ (*).$$

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[*] We must observe that in thus determining the constant, we no longer regard it as arbitrary; since by making $x=0$, in the equation (12), the constant is necessarily equal to the logarithm of a. Where the constant acquired this determinate value was, when, instead of $\int\frac{dx}{a+x}$, we put $\log\,(a+x)$; for the equation (11) shows us that $\frac{dx}{a+x}$ is, generally, the differential of... $\text{C}+\frac{x}{a}-\frac{x^2}{2a^2}+\frac{x^3}{3a^3}-\&\text{c.}$; but the series $\log a+\frac{x}{a}-\frac{x^2}{2a^2}+\&\text{c.}$, which is the development of $\log\,(a+x)$, is a particular case of the preceding series, the case, viz., in which $\text{C}=\log a$. Thus, the putting $\log\,(a+x)$ in place of

285. As a second example, let us take the series $\frac{dx}{1+x^2}$.

This differential being written thus $\frac{1}{1+x^2}\times dx$, we must find the development of $\frac{1}{1+x^2}$; for which purpose, comparing our expression with $\frac{1}{1-z}$, we shall have $z=-x^2$, and substituting this value in equation (10), we shall find

$$\frac{1}{1+x^2}=1-x^2+x^4-x^6+\text{ \&c.} \dots (13);$$

and, therefore,

$$\int\frac{dx}{1+x^2}=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\text{ \&c.} \dots +C,$$

or, art. 276,

$$\tan^{-1}x=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\text{ \&c.} \dots +C \dots (14).$$

When $x=0$, the arc becomes 0, and we have therefore $C=0$.

286. If the tangent be greater than unity, the terms of this series will go on increasing, and we cannot therefore give an approximate value of the arc; in this case, we shall obtain a converging series by putting $x=\frac{1}{x}$ in the equation (13), which will change it into

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$\int\frac{dx}{a+x}$ was as if, of all the series which are the integral of $\frac{dx}{a+x}$, we had chosen that in which the constant is equal to $\log a$.

This remark will apply to the rest of the expressions which we shall integrate by series.

$$\frac{1}{1+\frac{1}{x^2}}=1-\frac{1}{x^2}+\frac{1}{x^4}-\frac{1}{x^6}+\&c.$$

whence, multiplying the two terms of the first side by x^2, we shall have

$$\frac{x^2}{1+x^2}=1-\frac{1}{x^2}+\frac{1}{x^4}-\frac{1}{x^6}+\&c.$$

and dividing by x^2, we shall obtain

$$\frac{1}{x^2+1}=\frac{1}{x^2}-\frac{1}{x^4}+\frac{1}{x^6}-\frac{1}{x^8}+\&c.\ (*);$$

consequently,

$$\int\frac{dx}{1+x^2}=\int\left(\frac{1}{x^2}-\frac{1}{x^4}+\frac{1}{x^6}-\frac{1}{x^8}+\&c.\right)dx;$$

and, performing the integrations separately,

$$\tan^{-1}x=-\frac{1}{x}+\frac{1}{3x^3}-\frac{1}{5x^5}+\&c.+C \ldots . (15).$$

To determine the value of the constant, we must not put $x=0$, for then the terms of the second side of equation (15) will become infinite; but by making $x=\infty$, the expression $\text{tang}^{-1}x$ will become equal to the quadrant of the circle, and the equation will become *quadrant of circle* $=0+C$; whence, representing the quadrant by $\frac{1}{2}\pi$, the equation (15) will give us

$$\text{tang}^{-1}x=\tfrac{1}{2}\pi-\frac{1}{x}+\frac{1}{3x^3}-\frac{1}{5x^5}+\&c.$$

287. To integrate by the series

$$\frac{dx}{\sqrt{1-x^2}}=(1-x^2)^{-\frac{1}{2}}dx,$$

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[*] We might arrive at once at the same result by dividing 1 by $1+x^2$.

we will develop $(1-x^2)^{-\frac{1}{2}}$ by the binomial theorem in the following manner: we will calculate first the coefficients of the developments of $(1-x^2)^m$, on the hypothesis of $m=-\frac{1}{2}$, by writing down in order

$$m,\ \frac{m-1}{2},\ \frac{m-2}{3},\ \frac{m-3}{4},\ \&c.,$$

and changing m into $-\frac{1}{2}$ in these expressions, when they will become

$$-\frac{1}{2},\ -\frac{3}{4},\ -\frac{5}{6},\ -\frac{7}{8},\ \&c.$$

and multiplying $-\frac{1}{2}$ by $-\frac{3}{4}$, that product by $-\frac{5}{6}$, and so on, we shall form the coefficients which are to be substituted in the place of A, B, C, &c. in the equation

$$(1-x^2)^{-\frac{1}{2}}=1-\mathrm{A}x^2+\mathrm{B}x^4-\mathrm{C}x^6+\ \&c.,$$

which will give

$$\frac{1}{\sqrt{1-x^2}}=1+\frac{1}{2}x^2+\frac{1}{2}\cdot\frac{3}{4}x^4+\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}x^6+\ \&c.,$$

and by integrating the equation

$$\frac{dx}{\sqrt{1-x^2}}=\left(1+\frac{1}{2}x^2+\frac{1}{2}\cdot\frac{3}{4}x^4+\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}x^6+\ \&c.\right)dx,$$

we shall find

$$\sin^{-1}x=x+\frac{1}{2}\cdot\frac{x^3}{3}+\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{x^5}{5}+\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{x^7}{7}+\ \&c.\ \ldots\ (16).$$

We put no constant, because when $x=0$ the arc whose sign is x vanishes.

288. There are cases in which, to determine the value of the constant, we must neither make $x=0$, nor $x=\infty$. For example, let the expression be

$$\frac{dx}{\sqrt{x^2-1}}=(x^2-1)^{-\frac{1}{2}}dx=(x^2)^{-\frac{1}{2}}\left(1-\frac{1}{x^2}\right)^{-\frac{1}{2}}dx=\frac{dx}{x}\left(1-\frac{1}{x^2}\right)^{-\frac{1}{2}};$$

by putting down in order

$$m,\ \frac{m-1}{2},\ \frac{m-2}{3},\ \&c.$$

and making

$$m=-\frac{1}{2},$$

we find

$$-\frac{1}{2},\ -\frac{3}{4},\ -\frac{5}{6},\ \&c.;$$

whence we conclude, as in art. 287, that

$$\frac{dx}{\sqrt{x^2-1}}=\frac{dx}{x}\left(1+\frac{1}{2}\cdot\frac{1}{x^2}+\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{x^4}+\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{1}{x^6}+\&c.\right);$$

and by integrating we shall find

$$\int\frac{dx}{\sqrt{x^2-1}}=\log x-\frac{1}{2}\cdot\frac{1}{2x^2}-\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{4x^4}-\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{1}{6x^6}-\&c.$$

On the other hand,

$$\int\frac{dx}{\sqrt{x^2-1}}=\int\frac{dx}{\sqrt{x^2-1}}\cdot\times\frac{x+\sqrt{x^2-1}}{x+\sqrt{x^2-1}}$$

$$=\int\frac{\frac{xdx}{\sqrt{x^2-1}}+dx}{x+\sqrt{x^2-1}}=\log\left(x+\sqrt{x^2-1},\right)$$

and therefore

$$\log\left(x+\sqrt{x^2-1}\right)=\log x-\frac{1}{2}\cdot\frac{1}{2x^2}-\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{4x^4}-\&c.\ .\ .\ .\ (17).$$

To determine the value of the constant, we must not put $x=\infty$, for then $\log x$ will become ∞; nor, on the other hand, must we put $x=0$, for the terms $\log x$, $\frac{1}{2}\cdot\frac{1}{2x^2}$, &c. will then become ∞; but if we suppose $x=1$, the equation (17) will become

$$0=0-\frac{1}{2}\cdot\frac{1}{2}-\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{4}-\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{1}{6}-\&c.+C=0,$$

which gives

$$C=\frac{1}{2}\cdot\frac{1}{2}+\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{4}+\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{1}{6}+\&c.$$

289. The formula (16) will serve to determine an approximate value of the circumference; for by making $x=\frac{1}{2}$, it is reduced to

$$\sin^{-1}\frac{1}{2}=\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{2^3}+\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{5}\cdot\frac{1}{2^5}+\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{1}{7}\cdot\frac{1}{2^7}+\&c.:$$

but the sine whose value is $\frac{1}{2}$, is equal to half the side of a regular hexagon, and answers to the twelfth part of the circumference; so that we shall have

$$\frac{circumference}{12}=\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{2^3}+\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{5}\cdot\frac{1}{2^5}+\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{1}{7}\cdot\frac{1}{2^7}+\&c.,$$

and consequently

$$circumference=12\left(\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{2^3}+\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{5}\cdot\frac{1}{2^5}+\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{1}{7}\cdot\frac{1}{2^7}+\&c.\right),$$

taking the ten first terms of the series

$$\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{2^3}+\&c.,$$

we shall find for the sum

$$0,52359877,$$

and therefore

$$\frac{1}{2}circumference=6(0,52359877)=3,14159262,$$

a value which is correct to the last figure of the decimals.

290. We have found, art. 284,

$$\log(a+x)=\log a+\frac{x}{a}-\frac{x^2}{2a^2}+\frac{x^3}{3a^3}-\&c.$$

Since this series is but slightly convergent, we will put . . . . $x=-x$, when we shall have

$$\log(a-x)=\log a-\frac{x}{a}-\frac{x^2}{2a^2}-\frac{x^3}{3a^3}-\&c.,$$

and this equation, being subtracted from the former one, will give us

$$\log(a+x)-\log(a-x)=2\left(\frac{x}{a}+\frac{x^3}{3a^3}+\frac{x^5}{5a^5}+\&c.\right)$$

or

$$\log\left(\frac{a+x}{a-x}\right)=2\left(\frac{x}{a}+\frac{x^3}{3a^3}+\frac{x^5}{5a^5}+\&c.\right)\ \ldots\ (18).$$

291. To determine, for example, the logarithm of 2, by this formula, we shall suppose

$$\frac{a+x}{a-x}=\frac{2}{1},$$

and consequently

$$a+x=2,\ a-x=1,$$

whence

$$a=\frac{3}{2},\ x=\frac{1}{2},\ \frac{x}{a}=\frac{1}{3},\ \frac{x^2}{a^2}=\frac{1}{9},\ \&c.,$$

and substituting, we shall have

$$\log 2=2\left(\frac{1}{3}+\frac{1}{3}\cdot\frac{1}{27}+\frac{1}{5}\cdot\frac{1}{243}+\&c.\right).$$

Limiting ourselves to the ten first terms of this series, reduced to decimals, we shall determine the value of the logarithm of 2; and taking the triple of this logarithm, we shall have that of 2^3 or 8. If, then, we calculate by the formula (18) the logarithm of $\frac{10}{8}$, and add to it that of 8, we shall have the logarithm of $\frac{10}{8}\times 8=\log 10$; and we see that by similar processes, the formula (18) will give every other logarithm;

but it must be observed that these logarithms belong to the Napierian system. To deduce from them the tabular logarithms, if we represent by La the tabular logarithm of a number a, we shall have $a=10^{\mathrm{L}a}$; taking the Napierian logarithms, this equation will give us

$$\log a=\log 10^{\mathrm{L}a}=\mathrm{L}a\log 10,$$

and consequently

$$\mathrm{L}a=\frac{\log a}{\log 10};$$

so that the tabular logarithm of any number is equal to the Napierian logarithm of that number, divided by the Napierian logarithm of 10.

292. A series has been found, for determining a logarithm, still more convergent than that given by the formula (18); it may be deduced from that formula in the following manner:

By dividing $a+x$ by $a-x$, we find

$$\frac{a+x}{a-x}=1+\frac{2x}{a-x};$$

representing the fraction $\frac{2x}{a-x}$ by $\frac{v}{z}$, we have the equation

$$\frac{a+x}{a-x}=1+\frac{v}{z}=\frac{z+v}{z};$$

multiplying by $a-x$, there results

$$a+x=a-x+\frac{av}{z}-\frac{vx}{z},$$

and transposing, we obtain

$$2x+\frac{vx}{z}=\frac{av}{z};$$

multiplying, then, by z, we find

$$2xz+vx=av,$$

and consequently

$$\frac{x}{a}=\frac{v}{2z+v};$$

substituting these values of $\frac{a+x}{a-x}$ and $\frac{x}{a}$ in the formula (18), we have this result

$$\log\left(\frac{z+v}{z}\right)=2\left(\frac{v}{2z+v}+\frac{v^3}{3(2z+v)^3}+\frac{v^5}{5(2z+v)^5}+\&c.\right),$$

and lastly,

$$\log(z+v)=\log z+2\left(\frac{v}{2z+v}+\frac{v^3}{3(2z+v)^3}+\frac{v^5}{5(2z+v)^5}+\&c.\right)$$

For example, to obtain the logarithm of 2, we must put $v=1$, $z=1$, and consequently $\log z=0$; when substituting in the preceding formula, we shall have

$$\log 2=2\left(\frac{1}{3}+\frac{1}{3.(3)^3}+\frac{1}{5(3)^5}+\&c.\right)$$

This logarithm must be divided by the Napierian logarithm of 10, art. 291, to obtain the tabular logarithm of 2.

On the method of rational fractions.

293. Let the expression proposed for integration be

$$\frac{Px^m+Qx^{m-1}\ldots\ldots+Rx+S}{P'x^n+Q'x^{n-1}\ldots\ldots+R'x+S'}dx,$$

in which the multiplier of dx is a rational fraction; we will show that in the given expression, we may always suppose that n is greater than m; for if this should not be the case the integration may be reduced to that of a differential of the same form, in which the highest power of x in the denominator is greater than the highest power of x in the numerator, by simply dividing the numerator by the denominator, as in the following example:

Let the expression be

$$\frac{Px^3+Qx^2+Rx+S}{Q'x^2+R'x+S'},$$

dividing first by Q', we shall have

$$\frac{\frac{P}{Q'}x^3+\frac{Q}{Q'}x^2+\frac{R}{Q'}x+\frac{S}{Q'}}{x^2+\frac{R'}{Q'}x+\frac{S'}{Q'}};$$

making then

$$\frac{P}{Q'}=P'',\quad \frac{Q}{Q'}=Q'',\quad \frac{R}{Q'}=R'',\quad \frac{S}{Q'}=S'',$$

$$\frac{R'}{Q'}=R''',\quad \frac{S'}{Q'}=S''',$$

we shall have

$$\frac{P''x^3+Q''x^2+R''x+S''}{x^2+R'''x+S'''};$$

and the division will be effected in the following manner:

$$\begin{array}{l|l|l} x^2+R'''x+S''' & P''x^3+Q''x^2+R''x+S'' & P''x+M \\ & P''x^3+R'''P''x^2+S'''P''x & \\ \hline \text{1}^{\text{st}}\text{ rem}^{\text{r}}\ldots\ldots & (Q''-R'''P'')x^2+(R''-S'''P'')x+S'' & \\ \quad = & Mx^2+Nx+S'' & \\ & Mx^2+MR'''x+MS''' & \\ \hline \text{2}^{\text{nd}}\text{ rem}^{\text{r}}\ldots\ldots\ldots\ldots & (N-MR''')x+S''-MS''' & \end{array}$$

This last remainder may be represented by $Kx+L$, and then we have

$$\frac{Px^3+Qx^2+Rx+S}{Q'x^2+R'x+S'}dx=(P'x+M)dx+\frac{Kx+L}{x^2+R'''x+S'''}dx;$$

and by integrating, we obtain

$$\int\frac{Px^3+Qx^2+Rx+S}{Q'x^2+R'x+S'}dx=\frac{P''x^2}{2}+Mx+\int\frac{Kx+L}{x^2+R'''x+S'''}dx+C;$$

thus the question reduces itself to integrating

$$\frac{Kx+L}{x^2+R'''x+S'''}dx.$$

294. It follows from this that, whatever be the rational fraction under consideration, its integration may always be reduced, in the most general case, to that of

$$\frac{Px^{n-1}+Qx^{n-2}+\ldots+Rx+S}{P'x^{n}+Q'x^{n-1}+\ldots+R'x+S'}dx.$$

Considering the denominator of this fraction as the product of n factors, such as $x-a$, $x-b$, $x-c$, &c., these factors may be real or imaginary, equal or unequal.

To commence with the most simple case, we will suppose them real and unequal, and we must then proceed as in the following examples:

295. Let it be proposed first to integrate $\frac{adx}{x^2-a^2}$: resolving the denominator into its factors, we shall have

$$\frac{adx}{x^2-a^2}=\frac{adx}{(x-a)(x+a)};$$

and we will suppose

$$\frac{adx}{(x-a)(x+a)}=\left(\frac{A}{x-a}+\frac{B}{x+a}\right)dx \ldots . (19),$$

where A and B are constants which we must determine. For this purpose, reducing the second side of the equation to a common denominator, we shall obtain

$$\frac{adx}{(x-a)(x+a)}=\frac{(Ax+Aa+Bx-Ba)}{(x-a)(x+a)}dx;$$

and suppressing the common divisor $(x-a)(x+a)$ and the factor dx, there will remain

$$a=Ax+Aa+Bx-Ba \ldots . (20);$$

and, arranging according to x, we shall have

$$(A+B)x+(A-B-1)a=0.$$

Now x being indeterminate, as the proposed differential necessarily supposes*, this equation must hold good whatever x be;

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* In fact, the characteristic d, which precedes x, intimates that x is considered as variable.

and, consequently, according to the method of indeterminate coefficients, the coefficients of the different powers of x must be separately equated to zero; or, which comes to the same thing, the terms must be equated to each other, which, in equation (20), contain the same powers of x, when we shall have

$$A+B=0,\ (A-B-1)\,a=0,$$

and these equations give

$$A=\frac{1}{2},\ B=-\frac{1}{2}.$$

Substituting, therefore, these values in equation (19), we shall have

$$\int\frac{a\,dx}{x^2-a^2}=\frac{\frac{1}{2}dx}{x-a}-\frac{\frac{1}{2}dx}{x+a},$$

and by integrating, we shall find

$$\int\frac{a\,dx}{x^2-a^2}=\tfrac{1}{2}\log(x-a)-\tfrac{1}{2}\log(x+a)+C,$$

and, consequently,

$$\int\frac{a\,dx}{x^2-a^2}=\tfrac{1}{2}\log\frac{x-a}{x+a}+C=\log\left(\frac{x-a}{x+a}\right)^{\frac{1}{2}}+C.$$

For a second example, let us take the fraction $\frac{a^3+bx^2}{a^2x-x^3}dx$: the factors of the denominator are x and a^2-x^2, and since a^2-x^2 resolves itself into $(a-x)\times(a+x)$, the simple factors of the denominator are x, $a-x$, and $a+x$; and the expression to be integrated therefore is

$$\frac{a^3+bx^2}{x(a-x)(a+x)}dx:$$

assume

$$\frac{a^3+bx^2}{x(a-x)(a+x)}=\frac{A}{x}+\frac{B}{a-x}+\frac{C}{a+x}\ .\ .\ .\ .\ (21);$$

then reducing these fractions to a common denominator, there results

$$\frac{a^3+bx^2}{x(a-x)\,(a+x)}=\frac{Aa^2-Ax^2+Bax+Bx^2+Cax-Cx^2}{x(a-x)\,(a+x)},$$

and equating the coefficients of the same powers of x, we shall have

$$B-A-C=b,\ Ba+Ca=0,\ Aa^2=a^3.$$

The last of these equations gives us $A=a$, which reduces the first to $B-C=a+b$; and this equation being combined with the second, we obtain

$$B=\frac{a+b}{2},\ C=-\frac{a+b}{2};$$

putting these values of A, B, C, in equation (21), we find

$$\frac{a^3+bx^2}{a^2x-x^3}dx=\frac{adx}{x}+\frac{a+b}{2(a-x)}dx-\frac{a+b}{2(a+x)}dx;$$

and therefore, by integrating,

$$\int\frac{a^3+bx^2}{a^2x-x^3}dx=a\log x-\frac{a+b}{2}\log(a-x)\,{}^{*}$$
$$-\frac{a+b}{2}\log(a+x)+C$$

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[*] To take the integral of $\frac{a+b}{2(a-x)}dx$, since the differential of $a-x$ is $-dx$, we must put the differential under the form

$$-\frac{a+b}{2}\times-\frac{dx}{a-x},$$

and we see that the integral will be

$$-\frac{a+b}{2}\log(a-x)+C.$$

$$= a \log x - \frac{(a+b)}{2}[\log (a-x) + \log (a+x)] + C$$

$$= a \log x - \frac{(a+b)}{2}\log (a-x)(a+x) + C$$

$$= a \log x - \frac{(a+b)}{2}\log (a^2-x^2) + C$$

$$= a \log x - (a+b)\log (a^2-x^2)^{\frac{1}{2}} + C.$$

296. For a third example, let the fraction be $\frac{3x-5}{x^2-6x+8}dx$. Since we have first to resolve the denominator into factors of the first degree, we will observe that if we have an equation of the same form, represented by $z^2-6z+8=0$, and it be satisfied by the values $z=2$ and $z=4$, we may conclude, thence, that it is equal to the product $(z-2)\ (z-4)=0$. But, by performing the multiplication, we see that whatever be the value which we give to z, the product will always be z^2-6z+8; and therefore, when instead of z we put x, we shall have still

$$(x-2)\ (x-4) = x^2-6x+8\,;$$

consequently, whatever be the value of the polynomial . . . . x^2-6x+8, it may be resolved into its factors as though it were equal to zero.

Having therefore found that the roots of the equation $x^2-6x+8=0$ are 2 and 4, we may assume

$$\frac{3x-5}{x^2-6x+8}dx = \frac{Adx}{x-2} + \frac{Bdx}{x-4}\ \ .\ .\ .\ (22),$$

and suppressing the common factor dx, which we shall always do in future, we shall find, after reducing to a common denominator,

$$\frac{3x-5}{x^2-6x+8} = \frac{Ax-4A+Bx-2B}{x^2-6x+8}\ ;$$

equating, then, the coefficients of the same powers of x, we shall obtain these equations of condition,

$$-5=-4A-2B,\ 3=A+B,$$

whence we shall deduce

$$B=\tfrac{7}{2},\ A=-\tfrac{1}{2};$$

and putting these values in the equation (22), we shall find

$$\int\frac{3x-5}{x^2-6x+8}=-\frac{1}{2}\int\frac{dx}{x-2}+\frac{7}{2}\int\frac{dx}{x-4}+C$$
$$=\frac{7}{2}\log(x-4)-\frac{1}{2}\log(x-2)+C.$$

297. As another example let us take

$$\frac{xdx}{x^2+4ax-b^2}:$$

equating the denominator to zero, and resolving the equation, we find

$$x^2+4ax-b^2=(x+2a+\sqrt{4a^2+b^2})\,(x+2a-\sqrt{4a^2+b^2});$$

representing this last product more simply by

$$(x+K)\,(x+L),$$

we will suppose now

$$\frac{x}{x^2+4ax-b2}=\frac{A}{x+K}+\frac{B}{x+L},$$

and reducing the second side to a common denominator, we shall find

$$\frac{x}{x^2+4ax-b^2}=\frac{Ax+AL+Bx+BK}{x^2+4ax-b^2};$$

whence we deduce

$$A+B=1,\ AL+BK=0$$

consequently

$$A=\frac{K}{K-L},\quad B=-\frac{L}{K-L};$$

and therefore

$$\int\frac{x\,dx}{x^2+4ax-b^2}=\frac{K}{K-L}\log(x+K)-\frac{L}{K-L}\log(x+L)+C.$$

298. In general, let

$$\frac{Px^{m-1}+Qx^{m-2}\ \ldots\ +Rx+S}{x^m+Q'x^{m-1}\ \ldots\ +R'x+S'}dx$$

be a rational fraction, in which the simple factors of the denominator are supposed unequal: to integrate it, we shall first resolve the equation

$$x^m+Q'x^{m-1}+\ \ldots\ +R'x+S=0,$$

and having found that it is the product of the factors, $x-a$, $x-b$, $x-c$, &c, assume

$$\frac{Px^{m-1}+Qx^{m-2}\ \ldots\ +Rx+S}{x^m+Qx^{m-1}\ \ldots\ +R'x+S'}=\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}+\&c.$$

Reducing, then, the second side of this equation to a common denominator, the numerator of each of these fractions will be multiplied by the product of the denominators of all the others, i. e. by a polynomial in x of the order $m-1$; and the second side of the equation will consequently be a polynomial consisting of m terms. It follows, therefore, that if we equate the coefficients of the same powers of x on the two sides of the equation, we shall have m equations of condition for determining the m coefficients, A, B, C, &c.; and these coefficients being known, we shall then only have to integrate a series of terms such as

$$\frac{A\,dx}{x-a},\ \frac{B\,dx}{x-b},\ \&c.;$$

and the integral required will therefore be

$$A\log(x-a)+B\log(x-b)+\&c.+\textit{constant}.$$

299. The method which we have followed, when the roots of the denominator are unequal, will not serve if, among the roots which we will still suppose to be real, some of them are equal. For we have seen that, on the hypothesis of the roots being unequal, we may assume

$$\frac{Px^4+Qx^3+Rx^2+Sx+T}{(x-a)(x-b)(x-c)(x-d)(x-e)}=\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}+\frac{D}{x-d}+\frac{E}{x-e}.$$

But if several of these roots should be equal, if, for instance, we should have $a=b=c$, the preceding equation would become

$$\frac{Px^4+\&c.}{(x-a)^3(x-d)(x-e)}=\frac{A+B+C}{x-a}+\frac{D}{x-d}+\frac{E}{x-e};$$

and then, on reducing the second side to a common denominator, $A+B+C$ might be considered as one constant A', and we see that the three constants A', D, and E would not be sufficient for establishing the five equations of condition which ought to be obtained by equating the coefficients of the same powers of x.

300. To obviate this difficulty, the fraction

$$\frac{Px^4+Qx^3+\&c.}{(x-a)^3(x-d)(x-e)}$$

must be decomposed into another series of fractions, such that, when reduced to a common denominator, they shall again produce the fraction.

Suppose, therefore,

$$\frac{Px^4+Qx^3+\&c.}{(x-a)^3\,(x-d)\,(x-e)}=\frac{A+Bx+Cx^2}{(x-a)^3}+\frac{D}{x-d}+\frac{E}{x-e};$$

then, reducing the second side of this equation to a common denominator, we shall have a polynomial in x of the fourth degree, which will contain five arbitrary constants; and which will consequently be sufficient for establishing the identity of the terms affected by the same powers of x.

We will now show that the term

$$\frac{A+Bx+Cx^2}{(x-a)^3}$$

may be put under the form

$$\frac{A'}{(x-a)^3}+\frac{B'}{(x-a)^2}+\frac{C'}{(x-a)},$$

A′, B′, C′, being indeterminate constants. To prove this, let

$$x-a=z;$$

we have then

$$x=a+z;$$

and, therefore,

$$\frac{A+Bx+Cx^2}{(x-a)^3}=\frac{A+Ba+Ca^2+Bz+2Caz+Cz^2}{z^3}$$

$$=\frac{A+Ba+Ca^2}{z^3}+\frac{B+2Ca}{z^2}+\frac{C}{z},$$

when, putting the value of z in this equation, we shall obtain

$$\frac{A+Bx+Cx^2}{(x-a)^3}=\frac{A+Ba+Ca^2}{(x-a)^3}+\frac{B+2Ca}{(x-a)^2}+\frac{C}{x-a},$$

a result of the proposed form, since A′, B′, C′, are any constants.

This demonstration will apply to an equation of any higher degree, and we therefore conclude that we may suppose, generally,

$$\frac{Px^{m-1}+Qx^{m-2}+\ldots Rx+S}{(x-a)^m}=\frac{A}{(x-a)^m}+\frac{A'}{(x-a)^{m-1}}$$
$$+\frac{A''}{(x-a)^{m-2}}\ldots+\frac{A''}{x-a}.$$

It follows from this, that to integrate the expression

$$\frac{Px^4+\&c.}{(x-a)^3(x-d)(x-e)}dx,$$

we must suppose

$$\frac{Px^4+\&c.}{(x-a)^3(x-d)(x-e)}$$
$$=\frac{A}{(x-a)^3}+\frac{A'}{(x-a)^2}+\frac{A''}{(x-a)}+\frac{D}{(x-d)}+\frac{E}{x-e};$$

the fractions then being reduced to a common denominator, we shall determine the constants A, A′, A″, D, E, by the process which we have already employed, and we shall have then to find the integrals of the following expressions:

$$\frac{E}{x-e}dx,\quad\frac{D}{x-d}dx,\quad\frac{A''dx}{x-a},\quad\frac{A'dx}{(x-a)^2},\quad\frac{Adx}{(x-a)^3}.$$

The three first are integrated by logarithms; in respect to the other two, since dx is the differential of the expression $x-a$, contained within the brackets, we may assume $x-a=z$, (art. 270), and we shall have

$$\int\frac{A'dx}{(x-a)^2}=\int\frac{A'dz}{z^2}=\int A'z^{-2}dz=-\frac{A'}{z}=-\frac{A'}{x-a},$$
$$\int\frac{Adx}{(x-a)^3}=\int\frac{Adz}{z^3}=\int Az^{-3}dz=-\frac{A}{2z^2}=-\frac{A}{2(x-a)^2};$$

and consequently

$$\int\frac{\mathrm{P}x^4+\mathrm{Q}x^3+\&c.}{(x-a)^3\,(x-d)\,(x-e)}=-\frac{\mathrm{A}'}{x-a}-\frac{\mathrm{A}}{2(x-a)^2}\ldots$$

$$\ldots+\mathrm{D}\log(x-d)+\mathrm{E}\log(x-e)+\textit{constant}.$$

301. Let us take, for example, the fraction

$$\frac{2axdx}{(x+a)^2};$$

we shall have

$$\frac{2ax}{(x+a)^2}=\frac{\mathrm{A}}{(x+a)^2}+\frac{\mathrm{A}'}{x+a};$$

reducing the second side to a common denominator, and suppressing it on both sides of the equation, there will remain

$$2ax=\mathrm{A}+\mathrm{A}'x+\mathrm{A}'a;$$

whence we shall deduce the equations of condition,

$$2a=\mathrm{A}',\ \mathrm{A}+\mathrm{A}'a=0;$$

which give

$$\mathrm{A}'=2a,\text{ and }\mathrm{A}=-2a^2;$$

and consequently

$$\frac{2axdx}{(x+a)^2}=-\frac{2a^2dx}{(x+a)^2}+\frac{2adx}{(x+a)}\ldots(23).$$

To obtain the integral, we must observe, that dx being the differential of $(x+a)$, we may suppose $x+a=z$; and therefore

$$\int\frac{2axdx}{(x+a)^2}=-2a^2\int\frac{dz}{z^2}+2a\int\frac{dz}{z};$$

when, integrating the first fraction by the rule of art. 262, and the other by logarithms, we shall obtain

$$\int\frac{2axdx}{(x+a)^2}=\frac{2a^2}{z}+2a\log z+\mathrm{C};$$

and replacing the value of z,

$$\int \frac{2ax\,dx}{(x+a)^2} = \frac{2a^2}{a+x} + 2a\log(a+x) + C.$$

302. As a second example, we will investigate the integral of

$$\frac{x^2dx}{x^3-ax^2-a^2x+a^3};$$

for this purpose, equating the denominator to zero, we see that the terms all destroy each other, on the hypothesis of $x=a$; and therefore the equation $x^3-ax^2-a^2x+a^3$ is divisible by $x-a$. Performing this division, we find for the quotient x^2-a^2; and thus the quantity to be integrated is

$$\frac{x^2dx}{(x^2-a^2)(x-a)} = \frac{x^2dx}{(x+a)(x-a)(x-a)}$$
$$= \frac{x^2dx}{(x-a)^2(x+a)}.$$

Assume, therefore,

$$\frac{x^2}{(x-a)^2(x+a)} = \frac{A}{(x-a)^2} + \frac{A'}{x-a} + \frac{B}{x+a} \;.\;.\;.\;.\;.\; (24);$$

then reducing the second side to a common denominator, we obtain

$$\frac{x^2}{(x-a)^2(x+a)} = \frac{A(x+a)+A'(x^2-a^2)+B(x-a)^2}{(x-a)^2(x+a)};$$

and, developing and equating the coefficients of the same powers of x, we get the equations of condition

$$A'+B=1,\quad A-2Ba=0,\quad Aa-A'a^2+Ba^2=0 \;.\;.\;.\;. (25).$$

Multiplying the first of these by a^2, and adding it to the third, we shall have

$$Aa+Ba^2=a^2;$$

this, in its turn, being added to the second multiplied by a, we find

$$a^2=2\mathrm{A}a, \text{ and } \mathrm{A}=\tfrac{1}{2}a\,;$$

and putting this value of A in the second of equations (25), we obtain

$$\mathrm{B}=\tfrac{1}{4}:$$

the first therefore gives

$$\mathrm{A}'=1-\tfrac{1}{4}=\tfrac{3}{4}\,;$$

and, by means of these values of the constants, the equation (24), multiplied by dx, becomes

$$\frac{x^2dx}{(x-a)^2(x+a)}=\frac{adx}{2(x-a)^2}+\frac{3dx}{4(x-a)}+\frac{dx}{4(x+a)}.$$

To integrate $\frac{adx}{2(x-a)^2}$, we must put $x-a=z$, when the expression will become

$$\frac{adz}{2z^2}=\frac{az^{-2}dz}{2},$$

and we shall have for the integral, art. 262,

$$\frac{-az^{-1}}{2}=-\frac{a}{2z}=-\frac{a}{2(x-a)}\,;$$

and therefore

$$\int\frac{x^2dx}{(x-a)^2(x+a)}=-\frac{a}{2(x-a)}+\frac{3}{4}\log(x-a).$$
$$+\frac{1}{4}\log(x+a)+constant.$$

303. We shall proceed in the same manner, if the denominator contain several groups of equal roots. For example, let the expression be

$$\frac{adx}{(x^2-1)^2}=\frac{adx}{(x-1)^2(x+1)^2}\,;$$

we shall assume

$$\frac{a}{(x-1)^2(x+1)^2}=\frac{A}{(x-1)^2}+\frac{A'}{(x-1)}+\frac{B}{(x+1)^2}+\frac{B'}{x+1}\cdot(26);$$

when, reducing to a common denominator, we shall find

$$\frac{a}{(x-1)^2(x+1)^2}=$$

$$\frac{A(x+1)^2+A'(x-1).(x+1)^2+B(x-1)^2+B'(x-1)^2(x+1)}{(x-1)^2(x+1)^2};$$

and suppressing the common denominators, and developing the numerators, we shall obtain the equations of condition

$$\begin{aligned}A'+B'&=0,\\ A+A'+B-B'&=0,\\ 2A-A'-2B-B'&=0,\\ A-A'+B+B'&=a.\end{aligned}$$

The first of these equations reduces the third to $2A-2B=0$, and therefore $A=B$; the second reduces the fourth to $2A+2B=a$, or $4A=a$; whence $A=\frac{a}{4}=B$; the fourth equation consequently becomes $B'-A'=\frac{1}{2}a$, and this being combined with the first, we find

$$A'=-\frac{a}{4},\ B'=\frac{a}{4}:$$

when, by means of these values of the constants, the differential proposed becomes

$$\frac{1}{4}a\left\{\frac{dx}{(x-1)^2}+\frac{dx}{(x+1)^2}-\frac{dx}{x-1}+\frac{dx}{x+1}\right\}.$$

The two first of these expressions must be integrated by the rules of articles (270) and (262), and the others by logarithms, when we shall find

$$\int\frac{adx}{(x^2-1)^2}=\frac{1}{4}a\left\{-\frac{1}{x-1}-\frac{1}{x+1}-\log(x-1)+\log(x+1)\right\}+C.$$

304. Before we proceed to examine the case in which the denominator contains imaginary roots, we will make a few observations on quantities of this description; and first, we will consider the equation

$$x^2+px+q=0\ .\ .\ .\ .\ (27),$$

and investigate the conditions necessary, in order that the roots of this equation may be imaginary: for this purpose, resolving the equation, we find

$$x=-\frac{1}{2}p\pm\sqrt{\frac{p^2}{4}-q},$$

and the first condition necessary that this value of x may be imaginary, is that the last term of the equation (27) be positive; for if it be negative, the expression $-q$, which is under the radical sign, will change its sign, and the surd part then involving only positive quantities, x cannot be imaginary: this condition being fulfilled, x will be imaginary, if q be greater than $\frac{1}{4}p^2$. The excess of q over $\frac{1}{4}p^2$ being then essentially positive, we will represent it by β^2, since a square is always positive; and we shall have

$$q=\frac{1}{4}p^2+\beta^2;$$

making $\frac{1}{4}p^2=\alpha^2$, for the avoiding of fractions, this equation will become

$$q=\alpha^2+\beta^2,$$

and substituting these values of p and q in the proposed equation, we shall find

$$x^2+2\alpha x+\alpha^2+\beta^2=0\ .\ .\ .\ (28);$$

an equation which, being solved, gives

$$x=-\alpha\pm\beta\sqrt{-1}\ .\ .\ .\ (29);$$

and its two roots are therefore

$$-\alpha+\beta\sqrt{-1},\ -\alpha-\beta\sqrt{-1},$$

which shows that its roots are disposed in pairs, so that one being known, the other will be given by changing the sign of the imaginary part.

305. Generally, an equation may have several pairs of imaginary roots, and each pair will give rise to a factor of the second degree, of the form

$$x^2+2\alpha x+\alpha^2+\beta^2\ .\ .\ .\ (30).$$

306. The imaginary roots are sometimes equal, excepting as to the sign; this happens when $\alpha=0$, and one of the roots is then $+\beta\sqrt{-1}$, the other $-\beta\sqrt{-1}$, and the factor of the second degree is reduced to $x^2+\beta^2$.

307. To give an example of an equation whose roots are imaginary, we will take the equation

$$x^2-6ax+10a^2=0;$$

resolving it, we find

$$x=3a\pm\sqrt{-a^2}=3a\pm a\sqrt{-1};$$

and comparing this value of x with equation (29), we have

$$-\alpha=3a,\ \beta=a;$$

in the present case, therefore, the equation (30) becomes

$$x^2-6ax+9a^2+a^2.$$

308. To conclude, when we have an equation such as

$$x^2+4x+12=0,$$

whose roots are imaginary *, we may compare it immediately

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* This will be recognised by the conditions of art. 304 being fulfilled.

with the formula (30), and we shall have $2\alpha=4$, and therefore $\alpha^2=4$; subtracting 4 from 12, there remains 8 for the value of β^2, and the equation may be put under the form

$$x^2+4x+4+8=0.$$

The term 8 is not, in fact, a perfect square; but we may consider it as the square of $\sqrt{8}$.

309. We will employ ourselves now in the integration of rational fractions, the denominators of which contain imaginary factors; and to commence with the most simple case, we will consider that in which we have only a pair of imaginary roots in the denominator; suppose, for instance, that having decomposed the denominator into its factors, we have found

$$\frac{P+Qx+Rx^2+Sx^3+\&c.}{(x-a)(x-b)\ldots.(x-h)(x^2+2\alpha x+\alpha^2+\beta^2)}dx;$$

we shall equate this fraction, as we have done before, art. 300, to the series of terms

$$\frac{Adx}{x-a}+\frac{Bdx}{x-b}\ldots.+\frac{Hdx}{x-h}+\frac{Mx+N}{x^2+2\alpha x+\alpha^2+\beta^2}dx;$$

and having determined the constants A, B, ... H, M, N, by the process already employed, all these terms, except the last, will be integrated by logarithms; in respect to the last, it will be integrated in the manner following:

The quantity $x^2+2\alpha x+\alpha^2$ being a perfect square, the term to be integrated may be written thus,

$$\frac{Mx+N}{(x+\alpha)^2+\beta^2}dx;$$

making $x+\alpha=z$, this becomes

$$\frac{Mz+N-M\alpha}{z^2+\beta^2}dz,$$

and representing the constant part $N-M\alpha$ by P, it is reduced to

$$\frac{Mz+P}{z^2+\beta^2}dz,$$

an expression which resolves itself into

$$\frac{Mzdz}{z^2+\beta^2}+\frac{Pdz}{z^2+\beta^2}.$$

To integrate the first, we must observe that zdz being the differential of $z^2+\beta^2$, with the exception of a constant factor, we may, art. 271, suppose $z^2+\beta^2=y$, which will give us, by differentiating,

$$zdz=\frac{dy}{2};$$

and substituting these values, we shall obtain $\frac{Mdy}{2y}$, whence the integral will be

$$\begin{aligned}\frac{M}{2}\log y=\frac{M}{2}\log(z^2+\beta^2)&=\frac{M}{2}\log[(x+\alpha)^2+\beta^2]\\&=\frac{M}{2}\log(x^2+2\alpha x+\alpha^2+\beta^2)\\&=M\log(x^2+2\alpha x+\alpha^2+\beta^2)^{\frac{1}{2}}\\&=M\log(\sqrt{x^2+2\alpha x+\alpha^2+\beta^2}.\end{aligned}$$

In regard to the expression $\frac{Pdz}{z^2+\beta^2}$, by dividing the two terms by β^2, it may be brought under the form

$$\frac{P}{\beta}\cdot\frac{\frac{dz}{\beta}}{\frac{z^2}{\beta^2}+1},$$

and we see that its integral is

$$\frac{\mathrm{P}}{\beta}\tan^{-1}\frac{z}{\beta}=\frac{\mathrm{N}-\mathrm{M}\alpha}{\beta}\tan^{-1}\frac{x+\alpha}{\beta};$$

lastly, therefore,

$$\int\frac{\mathrm{M}x+\mathrm{N}}{x^2+2\alpha x+\alpha^2+\beta^2}dx=\mathrm{M}\log\sqrt{x^2+2\alpha x+\alpha^2+\beta^2}+\frac{\mathrm{N}-\mathrm{M}\alpha}{\beta}$$

$$\tan^{-1}.\frac{x+\alpha}{\beta}\ \ldots\ (31).$$

310. Let us take, for example, the fraction $\frac{a+bx}{x^3-1}dx$; the denominator having $x-1$ for a factor, we shall find the other factor by division, and the fraction proposed may be put under the form

$$\frac{a+bx}{(x-1)\,(x^2+x+1)}dx,$$

when x^2+x+1 being the product of two imaginary factors, as may be seen by resolving the equation $x^2+x+1=0$, we shall assume

$$\frac{a+bx}{(x-1)\,(x^2+x+1)}=\frac{\mathrm{A}}{x-1}+\frac{\mathrm{M}x+\mathrm{N}}{x^2+x+1};$$

reducing to the same denominator, and proceeding as has been directed, we shall find

$$\mathrm{A}=\frac{a+b}{3},\ \mathrm{M}=-\frac{a+b}{3},\ \mathrm{N}=\frac{b-2a}{3};$$

the expression x^2+x+1 being then resolved into its simple factors, by comparing it with the expression (30), this will give us

$$2\alpha=1,\ \alpha^2+\beta^2=1,$$

and consequently

$$\alpha=\frac{1}{2},\ \beta=\sqrt{\frac{3}{4}};$$

substituting these values, and those of M and N in the equation (31), which gives us the second part of the integral, and observing that the first is

$$\int\frac{A\,dx}{x-1}=\frac{a+b}{3}\log(x-1),$$

we shall find

$$\int\frac{(a+bx)\,dx}{x^3-1}=\frac{a+b}{3}\log(x-1)-\frac{a+b}{3}\log\sqrt{x^2+x+1}$$
$$+\frac{b-a}{\sqrt{3}}\tan^{-1}\frac{x+\frac{1}{2}}{\sqrt{\frac{3}{4}}}+C.$$

311. When the fraction has in its denominator equal imaginary factors, it will contain one or more factors of the second degree, of the form $(x^2+2\alpha x+\alpha^2+\beta^2)^p$, accordingly as it contains one or more groups of equal imaginary factors. The factor

$$(x^2+2\alpha x+\alpha^2+\beta^2)^p$$

will correspond to the series of terms

$$\frac{H+Kx}{(x^2+2\alpha x+\alpha^2+\beta^2)^p}+\frac{H'+K'x}{(x^2+2\alpha x+\alpha^2+\beta^2)^{p-1}}$$
$$+\frac{H''+K''x}{(x^2+2\alpha x+\alpha^2+\beta^2)^{p-2}}\;\cdots\;+\frac{H_{\prime}+K_{\prime}x}{x^2+2\alpha x+\alpha^2+\beta^2}\;\cdot\cdot\;(32);$$

and having proceeded in the same manner for the other groups of equal factors, we must determine the constants

$$H,\ K,\ H',\ K',\ H'',\ K'',\ \ldots\ H_{\prime},\ K_{\prime},\ \&c.,$$

as before.

Multiplying, then, by dx, we shall have only to integrate each term separately, which may always be done if we know how to integrate the first term of the series (32) multiplied by dx, since all the others are of the same form. For this purpose we shall put the term under the form

$$\frac{H+Kx}{[(x+\alpha)^2+\beta^2]^p}dx;$$

making $x+\alpha=z$, it will become

$$\frac{H-K\alpha+Kz}{(z^2+\beta^2)^p}dz;$$

and representing the constant part $H-K\alpha$ by M, we shall have to integrate

$$\frac{M+Kz}{(z^2+\beta^2)^p}dz,$$

a fraction which may be resolved into the two

$$\frac{Kzdz}{(z^2+\beta^2)^p}+\frac{Mdz}{(z^2+\beta^2)^p}.$$

To integrate the first, since zdz is the differential of $z^2+\beta^2$, except as to a constant factor, we shall suppose $z^2+\beta^2=y$ (art. 271), when we shall have $zdz=\frac{1}{2}dy$, and substituting, we shall obtain

$$\int\frac{Kzdz}{(z^2+\beta^2)^p}=\int\frac{1}{2}K.\frac{dy}{y^p}=\frac{1}{2}K\int y^{-p}dy=\frac{1}{2}K.\frac{y^{-p+1}}{1-p}$$
$$=\frac{1}{2}K\frac{(\beta^2+z^2)^{-p+1}}{1-p}=\frac{1}{2}.\frac{K}{1-p}\,\frac{1}{(z^2+\beta^2)^{p-1}}+C.$$

It remains now to integrate $\frac{Mdz}{(z^2+\beta^2)^p}$ or $M(z^2+\beta^2)^{-p}dz$ (33), to arrive at which integral, we shall deduce it (*note fourth*) from that of $\int(z^2+\beta^2)^p dz$, in the following manner:

To diminish the index p by unity, is to divide by $z^2+\beta^2$; consequently, multiplying at the same time by that quantity, we shall have the identical equation

$$(z^2+\beta^2)^p dz=(z^2+\beta^2)^{p-1}(z^2+\beta^2)dz;$$

performing the multiplication indicated on the second side, there will result

$$(z^2+\beta^2)^p dz=\beta^2(z^2+\beta^2)^{p-1}dz+(\beta^2+z^2)^{p-1}z^2dz,$$

and integrating, we shall have

$$\int(\beta^2+z^2)^p dz=\beta^2\int(z^2+\beta^2)^{p-1}dz+\int(\beta^2+z^2)^{p-1}z^2dz \;.\;.\;.\; (34).$$

Of the two integrals on the second side of this equation, we shall leave the first under the integral sign; to the second we shall apply the method of parts. For this purpose, multiplying and dividing the expression $(\beta^2+z^2)^{p-1}z^2dz$ by 2, we shall put it under the form,

$$\frac{z}{2}(\beta^2+z^2)^{p-1}2zdz \;.\;.\;.\;.\; (35)\;;$$

then $(\beta^2+z^2)^{p-1}2zdz$ will be the differential of $\frac{(\beta^2+z^2)^p}{p}$, so that the expression (35) will become

$$\frac{z}{2}d.\frac{(\beta^2+z^2)^p}{p},$$

and comparing it with the formula

$$\int udv = uv-\int vdu,$$

we shall put

$$u=\frac{z}{2},\quad v=\frac{(\beta^2+z^2)^p}{p},$$

when we shall find

$$\int\frac{z}{2}(\beta^2+z^2)^{p-1}2zdz=\frac{z}{2}\frac{(\beta^2+z^2)^p}{p}-\int\frac{(\beta^2+z^2)^p}{p}\frac{dz}{2}.$$

Substituting this value in place of the last term of the equation (34), and putting the constants without the integral sign, the equation will become

$$\int(\beta^2+z^2)^p dz = \beta^2\int(\beta^2+z^2)^{p-1}dz$$
$$+\frac{z}{2}\frac{(\beta^2+z^2)^p}{p} - \frac{1}{2p}\int(\beta^2+z^2)^p dz;$$

transposing the last term on the second side, and reducing, we shall find

$$\frac{1+2p}{2p}\int(\beta^2+z^2)^p dz = \frac{z}{2}\frac{(\beta^2+z^2)^p}{p} + \beta^2\int(\beta^2+z^2)^{p-1}dz;$$

whence we deduce

$$\int(\beta^2+z^2)^{p-1}dz = -\frac{z}{2p\beta^2}(\beta^2+z^2)^p + \frac{1+2p}{2p\beta^2}\int(\beta^2+z^2)^p dz;$$

putting $p-1=-p$, and, consequently, $p=1-p$, we have, lastly,

$$\int(\beta^2+z^2)^{-p}dz = -\frac{z}{2(1-p)\beta^2}(\beta^2+z^2)^{-p+1}$$
$$+\frac{3-2p}{(2-2p)\beta^2}\int(\beta^2+z^2)^{-(p-1)}dz \;\dots\; (36).$$

By means of this formula, then, the integral of $(\beta^2+z^2)^{-p}dz$ is made to depend on that of $(\beta^2+z^2)^{-(p-1)}dz$, in which the numerical value of the index, instead of being p, will be less by *unity*; similarly the integral of $(\beta^2+z^2)^{-(p-1)}dz$ will be made to depend on that of $(\beta^2+z^2)^{-(p-2)}dz$, and so on; so that the index of the integral part being diminished by *unity* after each substitution, the expression to be integrated will at length become

$$(\beta^2+z^2)^{-1}dz = \frac{dz}{\beta^2+z^2};$$

and we have seen, art. 277, that the integral of this expression is

$$\frac{1}{\beta}\tan^{-1}\frac{z}{\beta}.$$

We do not attempt to make the integral of $(\beta^2+z^2)^{-1}dz$ dependent on that of $(\beta^2+z^2)^0dz$, a quantity which reduces itself to dz; for if in the formula (36) we should make $p=1$, the term

$$\frac{-z}{2(1-p)\beta^2}(\beta^2+z^2)^{-p+1}$$

would become infinite.

313. It follows from this theory that the integration of every rational fraction depends only on the three following formulæ:

$$1^{\circ}.\ \int x^m dx=\frac{x^{m+1}}{m+1}\,;\quad 2^{\circ}.\ \int\frac{dx}{x+a}=\log(x+a)\,;$$

$$3^{\circ}.\ \int\frac{dx}{x^2+a^2}=\frac{1}{a}\tan^{-1}\frac{x}{a}.$$

And it is therefore we say that every rational fraction may be integrated algebraically, or by logarithms, or by arcs of circles, or by the union of these methods.

314. We will conclude this theory by an example containing all the cases; let the rational fraction be therefore

$$\frac{Px^m+P'x^{m-1}+P''x^{m-2}+\&c.}{RR'R''\ldots SS'\ldots TT'\ldots UU'\ldots},$$

in which we have

$$\left.\begin{array}{l} R\ =x-a, \\ R'\ =x-b, \\ R''=x-c, \\ \ldots\ldots\ldots \end{array}\right\}\textit{factors real and unequal.}$$

$$\left.\begin{array}{l} S=(x-e)^m \\ S'=(x-d)^n \\ \ldots\ldots\ldots \end{array}\right\}\textit{factors real and equal.}$$

$$\left.\begin{array}{l} T\ =x^2+2\alpha x\ +\alpha^2\ +\beta^2 \\ T'=x^2+2\alpha' x+\alpha'^2+\beta'^2 \\ \ldots\ldots\ldots\ldots\ldots \end{array}\right\}\textit{factors imaginary and unequal.}$$

$$\left.\begin{array}{l} \mathrm{U}\ = (x^2+2\alpha_{,}x+\alpha_{,}^2+\beta_{,}^2)^p \\ \mathrm{U}' = (x^2+2\alpha_{,,}x+a_{,,}^2+\beta_{,,}^2)^p \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\}\ \textit{factors imaginary and equal.}$$

We shall assume then

$$\frac{\mathrm{P}x^m+\mathrm{P}'x^{m-1}+\mathrm{P}''x^{m-2}+\&c.}{\mathrm{RR'R''}\ldots\mathrm{SS'}\ldots\mathrm{TT'}\ldots\mathrm{UU'}\ldots}=\frac{\mathrm{A}}{x-a}+\frac{\mathrm{B}}{x-b}+\frac{\mathrm{C}}{x-c}\cdots$$

$$+\frac{\mathrm{E}}{(x-e)^m}+\frac{\mathrm{E}'}{(x-e)^{m-1}}+\frac{\mathrm{E}''}{(x-e)^{m-2}}\ \cdots\ +\ \&c.$$

$$+\frac{\mathrm{F}}{(x-f)^n}+\frac{\mathrm{F}'}{(x-f)^{n-1}}+\frac{\mathrm{F}''}{(x-f)^{n-2}}\ \cdots\ +\ \&c.$$

$$+\frac{\mathrm{G}+\mathrm{H}x}{x^2+2\alpha x+\alpha^2+\beta^2}+\frac{\mathrm{K}+\mathrm{L}x}{x^2+2\alpha' x+\alpha'^2+\beta'^2}+\&c.$$

$$+\frac{\mathrm{M}+\mathrm{N}x}{(x^2+2\alpha_{,}x+\alpha_{,}^2+\beta_{,}^2)^p}+\frac{\mathrm{M}'+\mathrm{N}'x}{(x^2+2\alpha_{,}x+\alpha_{,}^2+\beta_{,}^2)^{p-1}}+\&c.$$

$$+\frac{\mathrm{P}+\mathrm{Q}x}{(x^2+2\alpha_{,,}x+\alpha_{,,}^2+\beta_{,,}^2)^q}+\frac{\mathrm{P}'+\mathrm{Q}'x}{(x^2+2\alpha_{,,}x+\alpha_{,,}^2+\beta_{,,}^2)^{q-1}}+\&c.\,;$$

and having reduced to a common denominator, we shall proceed according to the rules already laid down and explained.

Integration of irrational functions.

315. When in a differential expression, which contains radicals, we can, by means of any transformation, make the radicals disappear, the integration will be reduced to that of rational fractions.

Radicals may always be made to disappear which affect only monomial quantities; the process to be employed for accomplishing this will be the same with the one we are about to make use of in the following example:

Let the expression be

$$\frac{\sqrt{x}-\frac{1}{3}a}{\sqrt[3]{x}-\sqrt{x}}\ \text{or}\ \frac{x^{\frac{1}{2}}-\frac{1}{3}a}{x^{\frac{1}{3}}-x^{\frac{1}{2}}}\,;$$

we shall first reduce the fractional indices to the same denominator, and having found that the common denominator is 6, we shall suppose $x=z^6$; we shall have then

$$\sqrt{x}=z^3,\ \sqrt[3]{x}=z^2,\ dx=6z^5dz\,;$$

and substituting these values, we shall find

$$\frac{\sqrt{x}-\frac{1}{3}a}{\sqrt[3]{x}-\sqrt{x}}dx=\frac{z^3-\frac{1}{3}a}{z^2-z^3}6z^5dz=\frac{6z^6-2az^3}{1-z}dz\,;$$

this expression we shall integrate by the method of rational fractions, and then substitute in the integral the value of z.

316. This method will not generally apply when the quantity under the radical sign is a polynomial; we may, however, integrate every expression in x, which contains $\sqrt{A+Bx+Cx^2}$, that is to say, every expression of the form

$$F(x,\sqrt{A+Bx+Cx^2})dx.$$

There may be two cases: the term Cx^2 may be positive, or it may be negative; if it be positive, we shall write the expression thus,

$$\sqrt{C}\sqrt{\frac{A}{C}+\frac{B}{C}x+x^2}\,;$$

but if that term be negative, we shall consider it as the product of $+C$ by $-x^2$, and then the radical may be put under the form,

$$\sqrt{C}\sqrt{\frac{A}{C}+\frac{B}{C}x-x^2},$$

and putting, for greater simplicity,

$$\frac{A}{C}=a,\ \frac{B}{C}=b,$$

we shall have to integrate the two expressions

$$F(x,\sqrt{a+bx+x^2})\,dx,\ \ F(x,\sqrt{a+bx-x^2})\,dx$$

We will now employ ourselves with the first.

Our aim being to obtain, by a transformation, the values of x, dx, and $\sqrt{a+bx+x^2}$, in a rational function of a new variable z, we will suppose

$$\sqrt{a+bx+x^2}=z+x\,^* \;.\;.\;.\;.\;.\; (37),$$

because on raising to the square, the terms in x^2 will destroy each other, and there will remain between z and x an equation of the first degree, from which we shall be able to deduce the values of x and dx in rational functions of z. Raising, therefore, equation (37) to the square, and suppressing the terms in x^2, we obtain

$$a+bx=2xz+z^2 \;.\;.\;.\;.\; (38),$$

whence we deduce

$$x=\frac{z^2-a}{b-2z} \;.\;.\;.\;.\; (39);$$

and by means of this value the equation (37) becomes

$$\sqrt{a+bx+x^2}=\frac{z^2-a}{b-2z}+z;$$

or, reducing to the same denominator,

$$\sqrt{a+bx+x^2}=-\frac{(z^2-bz+a)}{b-2z} \;.\;.\;.\;.\; (40).$$

It remains now to determine dx in terms of z, for which purpose differentiating the equation (38), we shall obtain

$$bdx=2xdz+2zdx+2zdz,$$

whence we shall deduce

$$(b-2z)dx=2(x+z)dz,$$

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[*] We might also equate the radical to $z-x$, because on squaring the two sides the terms in x^2 would equally vanish on either hypothesis.

and eliminating the radical betwixt the equation (37), and the equation (40), we shall have

$$x+z=-\frac{z^2-bz+a}{b-2z};$$

when, substituting this value in the preceding equation, we shall find

$$(b-2z)\,dx=-\frac{2(z^2-bz+a)}{b-2z}dz,$$

and therefore

$$dx=-\frac{2(z^2-bz+a)}{(b-2z)^2}dz \ldots (41).$$

317. Let us take, for example,

$$\frac{dx}{x\sqrt{A+Bx+Cx^2}}:$$

putting $\frac{A}{C}=a$, and $\frac{B}{C}=B$, the expression may be written thus,

$$\frac{dx}{\sqrt{C}\times x\sqrt{a+bx+x^2}}.$$

The equation (41), divided by the equation (40), will give us, after reduction,

$$\frac{dx}{\sqrt{a+bx+x^2}}=\frac{2dz}{b-2z},$$

and dividing by the equation (39), we shall have

$$\frac{dx}{x\sqrt{a+bx+x^2}}=\frac{2dz}{z^2-a};$$

multiplying the denominators by $\sqrt{C}$, this equation will become

$$\frac{dx}{\sqrt{C}\times x\sqrt{a+bx+x^2}} \text{ or } \frac{dx}{x\sqrt{A+Bx+Cx^2}}=\frac{2dz}{(z^2-a)\sqrt{C}},$$

a fraction integrable by the method of rational fractions, since $\sqrt{C}$ may be considered as an ordinary constant.

318. As a second example, we will take $dx\sqrt{m^2+x^2}$; comparing the radical part with that of the formula (40), we have $a=m^2$, $b=0$, and putting these values in the equations, (40) and (41), we shall find

$$\sqrt{m^2+x^2}=\frac{z^2+m^2}{2z},\quad dx=-\frac{z^2+m^2}{2z^2}dz,$$

whence

$$dx\sqrt{m^2+z^2}=-\frac{(z^2+m^2)^2}{4z}dz;$$

and having integrated this rational expression, we must substitute for z its value in terms of x.

319. The preceding method cannot be employed when Cx^2 is negative; for, on proceeding as above, we should have

$$\begin{aligned}\sqrt{A+Bx-Cx^2}&=\sqrt{C}\sqrt{\frac{A}{C}+\frac{B}{C}x-x^2}\\ &=\sqrt{C}\sqrt{a+bx-x^2},\end{aligned}$$

and if we should suppose $\sqrt{a+bx-x^2}=x+z$, on squaring the two sides of this equation the terms in x^2 would not vanish, but we should have a term $2x^2$, and the value of x in terms of z would result irrational. To treat this case, we must observe preliminarily that the polynomial $a+bx-x^2$ may always be resolved into real factors of the first degree*.

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* To demonstrate this, let the polynomial be written thus,

$$-(x^2-bx-a);$$

we shall find then the factors of x^2-bx-a by equating this expression to zero, which will give us

Let α and α' be the roots of the equation $x^2-bx-a=0$; we shall have then, from the property of equations,

$$x^2-bx-a=(x-\alpha)(x-\alpha'),$$

and, consequently, by changing the signs,

$$a+bx-x^2=-(x-\alpha)(x-\alpha')=(x-\alpha)(\alpha'-x);$$

substituting this value in the radical, we will suppose

$$\sqrt{(x-\alpha)(\alpha'-x)}=(x-\alpha)z \;.\;.\;.\;.\; (42),$$

which being squared, gives us

$$(x-\alpha)(\alpha'-x)=(x-\alpha)^2z^2,$$

and suppressing the common factor, we have

$$\alpha'-x=(x-\alpha)z^2 \;.\;.\;.\;.\; (43),$$

whence we deduce

$$x=\frac{\alpha'+\alpha z^2}{z^2+1},$$

therefore,

$$x-\alpha=\frac{\alpha'+\alpha z^2}{z^2+1}-\alpha,$$

and reducing to the same denominator,

$$x-\alpha=\frac{\alpha'-\alpha}{z^2+1}, \; (44),$$

$$x=\frac{b}{2}\pm\sqrt{\frac{b^2}{4}+a},$$

and, therefore, by the property of equations,

$$x^2-bx-a=\left(x-\frac{b}{2}+\sqrt{\frac{b^2}{4}+a}\right)\left(x-\frac{b}{2}-\sqrt{\frac{b^2}{4}+a}\right);$$

and since, by hypothesis, a represents a positive quantity, the factors which compose this product cannot be imaginary. Besides this, without resolving the equation $x^2-bx-a=0$, we may conclude, from the sign of its last term, that it has its roots real, (art. 304).

which value being substituted on the second side of the equation (42), we obtain

$$\sqrt{(x-\alpha)(\alpha'-x)} = \frac{\alpha'-\alpha}{z^2+1}z \; .\;.\;.\;. \; (45).$$

In regard to dx, we have only to differentiate the equation (44) to obtain its value in terms of z, and we shall have

$$dx = -\frac{2(\alpha'-\alpha)}{(z^2+1)^2}z\,dz \; .\;.\;.\;. \; (46).$$

320. Applying this process to the example

$$\frac{dx}{\sqrt{a+bx-x^2}},$$

we must divide the equation (46) by the equation (45), when we shall have

$$\frac{dx}{\sqrt{a+bx-x^2}} = -\frac{2(\alpha'-\alpha)z}{(z^2+1)^2 z\,.\,\dfrac{\alpha'-\alpha}{z^2+1}}.dz = -\frac{2dz}{z^2+1},$$

and, therefore,

$$\int\frac{dx}{\sqrt{a+bx-x^2}} = -2\tan^{-1}.z + \mathrm{C},$$

or, putting for z its value, given by equation (42),

$$\int\frac{dx}{\sqrt{a+bx-x^2}} = \mathrm{C} - 2\tan^{-1}.\sqrt{\frac{x-\alpha\cdot\alpha'-x}{x-\alpha}}$$

$$= \mathrm{C} - 2\tan^{-1}.\frac{\sqrt{\alpha'-x}}{\sqrt{x-\alpha}}.$$

321. Let us take also, for example, $dx\sqrt{2ax-x^2}$; comparing this radical with that of the equation (42), we shall have $\alpha=0$, $\alpha'=2a$, whence the equations (45) and (46) become

$$\sqrt{x(2a-x)}=\frac{2az}{z^2+1},\quad dx=-\frac{4az}{(z^2+1)^2}dz;$$

and these equations, being multiplied one by the other, give us

$$dx\sqrt{2ax-x^2}=-\frac{8a^2z^2dz}{(z^2+1)^2},$$

an expression which is integrable by the method of rational fractions.

Integration of binomial differentials.

322. We have seen that a method, very extensive in its application, for the integrating of irrational functions, is to transform the functions into others that are rational, so as to be able to apply the rules for rational fractions.

The difficulty is to determine the transformation which ought to be employed in each case; we have already stated the one that is applicable when the surds are trinomials, in which the variable does not rise above the second degree; and since expressions of this sort occur very frequently in analysis, the knowledge of the transformation necessary for rendering them rational will be of great service. We have also given a general process for rendering functions rational which contain only monomials raised to fractional powers; and we will now proceed to examine whether, by means of any transformation, binomials affected with fractional indices can be rendered rational.

323. The general form for binomials is

$$x^{m-1}(a+bx^n)^p dx\,*.$$

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* The binomial expression Ax^r+Bx^s being a particular case of the one $(Ax^r+Bx^s)^p$, it is to the latter form that we shall refer the binomial differentials: it may be written thus,

$$[x^r(A+Bx^{s-r})]^p=x^{rp}(A+Bx^{s-r})^p,$$

If p be a whole number, this formula will be integrated by art. 269; but when p is equal to a fraction $\frac{p}{q}$, we shall have

$$x^{m-1}(a+bx^n)^{\frac{p}{q}}dx \dots (47).$$

To render this expression rational, we will put

$$a+bx^n=z^q \dots \quad (48),$$

or, which comes to the same thing,

$$(a+bx^n)^{\frac{1}{q}}=z,$$

and, consequently,

$$(a+bx^n)^{\frac{p}{q}}=z^p \dots (49).$$

Then the equation (48) being differentiated, gives us

$$bnx^{n-1}dx=qz^{q-1}dz \dots (50);$$

the same equation being resolved in respect of x, we have

$$x=\left(\frac{z^q-a}{b}\right)^{\frac{1}{n}};$$

and therefore, raising the two sides of this equation to the power m, we obtain

$$x^m=\left(\frac{z^q-a}{b}\right)^{\frac{m}{n}};$$

differentiating the two sides, putting the constants in front, and dividing by m, we find

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and making $s-r=n$, $rp=m-1$, it becomes

$$x^{m-1}(A+Bx^n)^p.$$

We have replaced rp by $m-1$, rather than by m, because, as we shall see, the conditions of integrability are more readily expressed on that supposition.

$$x^{m-1}dx = \frac{q}{nb}\left(\frac{z^q - a}{b}\right)^{\frac{m}{n}-1} z^{q-1}dz\ ;$$

and substituting this value in the equation (47), as also that of $(a+bx^n)^{\frac{p}{q}}$ given by equation (49), we have lastly

$$\frac{q}{nb}\left(\frac{z^q-a}{b}\right)^{\frac{m}{n}-1} z^{q+p-1}dz \ \ldots\ldots (51).$$

This expression is rational when $\frac{m}{n}$ is a positive whole number, for then $\frac{z^q-a}{b}$ is raised to an integral power, and we may reduce the expression (51) to a limited number of monomial terms, each of which will be integrable by art. 262, or by art. 268. If $\frac{m}{n}$ be a negative whole number, the expression (51) becomes also rational, and may be integrated by the method of rational fractions.

324. Let us take for example the expression

$$x^5(a+bx^2)^{\frac{2}{3}}dx\ ;$$

in this case we shall have

$$p=2,\ q=3,\ m-1=5,\ \text{or}\ m=6,\ n=2,$$

and consequently the condition of integrability is satisfied. Substituting, therefore, these values in the expression (51), we shall have to integrate

$$\frac{3}{2b^3}(z^3-a)^2z^4dz = \frac{3z^{10}}{2b^3}dz - \frac{3a}{b^3}z^7dz + \frac{3a^2}{2b^3}z^4dz\ ;$$

whence

$$\int x^5(a+bx^2)^{\frac{2}{3}}dx = \frac{3z^{11}}{22b^3} - \frac{3az^8}{8b^3} + \frac{3a^2z^5}{10b^3} + C\ ;$$

and we must substitute in this result the value of z in terms of x.

325. To obtain another condition of integrability, we will write the expression (47) in the manner following:

$$x^{m-1}\left[\left(\frac{a}{x^n}+b\right)x^n\right]^{\frac{p}{q}}dx;$$

and, raising the factors of the product $\left(\frac{a}{x^n}+b\right)x^n$ to the power $\frac{p}{q}$, we shall have

$$x^{m-1}x^{\frac{np}{q}}\left(\frac{a}{x^n}+b\right)^{\frac{p}{q}}=x^{m+\frac{np}{q}-1}(ax^{-n}+b)^{\frac{p}{q}}dx.$$

But, according to the preceding demonstration, in order that this quantity may be integrable, we must have

$$\frac{m+\frac{np}{q}}{n}=\textit{whole number},$$

or, performing the division,

$$\frac{m}{n}+\frac{p}{q}=\textit{whole number}.$$

326. Let us take, for example, the expression $x^4dx\sqrt[3]{a+bx^3}$, writing it thus:

$$x^{5-1}(a+bx^3)^{\frac{1}{3}}dx,$$

we have

$$m=5,\ n=3,\ p=1,\ q=3,$$

consequently

$$\frac{m}{n}+\frac{p}{q}=\frac{5}{3}+\frac{1}{3}=2,$$

and the quantity, therefore, is integrable.

In this case we shall have, (art. 325),

$$x^{5-1}(a+bx^3)^{\frac{1}{3}}dx=x^{5-1}\left(\frac{a}{x^3}+b\right)^{\frac{1}{3}}x\,dx;$$

and adding together the indices of x, this expression will become

$$x^5(ax^{-3}+b)^{\frac{1}{3}}dx;$$

making $ax^{-3}+b=z^3$, we shall find

$$(ax^{-3}+b)^{\frac{1}{3}}=z,\ x^{-3}=\frac{z^3-b}{a},$$

or

$$\left(\frac{a}{x^3}+b\right)^{\frac{1}{3}}=z,\ \frac{1}{x^3}=\frac{z^3-b}{a};$$

the latter of these equations gives us

$$x^3=\frac{a}{z^3-b},$$

whence we deduce, by differentiation,

$$x^2dx=-\frac{az^2dz}{(z^3-b)^2};$$

multiplying the two last equations together, we have

$$x^5dx=-\frac{a^2z^2dz}{(z^3-b)^3};$$

and this value of x^5dx, and that of $(ax^{-3}+b)^{\frac{1}{3}}$ being substituted in the expression (52), we find, lastly,

$$x^5(ax^{-3}+b)^{\frac{1}{3}}dx = -\frac{a^2z^3dz}{(z^3-b)^3},$$

an expression which is integrable by the method of rational fractions.

Formulæ of reduction of binomial differentials.

327. When the equation $x^{m-1}dx\,(a+bx^n)^p$ does not satisfy the conditions of integrability which we have just laid down, we may apply to it the method of integration by parts, in the manner following:

Comparing the formula $\int x^{m-1}dx\,(a+bx^n)^p$ with the first side of the equation

$$\int udv = uv - \int vdu,$$

we may assume

$$(a+bx^n)^p = u,\ x^{m-1}dx = dv, \text{ and therefore } v = \frac{x^m}{m},$$

and we shall have, putting the constants without the sign of integration,

$$\int x^{m-1}dx(a+bx^n)^p = (a+bx^n)^p\frac{x^m}{m} - \frac{pnb}{m}\int x^m(a+bx^n)^{p-1}x^{n-1}dx,$$

or,

$$\int x^{m-1}dx(a+bx^n)^p = (a+bx^n)^p\frac{x^m}{m} - \frac{pnb}{m}\int x^{m+n-1}(a+bx^n)^{p-1}dx \quad (53);$$

on the other hand, we have the equation

$$(a+bx^n)^p = (a+bx^n)^{p-1}(a+bx^n);$$

and, multiplying out, this equation gives

$$(a+bx^n)^p = a(a+bx^n)^{p-1} + bx^n(a+bx^n)^{p-1};$$

whence multiplying the two sides by $x^{m-1}dx$, we find

$$\int x^{m-1}dx(a+bx^n)^p$$
$$= a\int x^{m-1}dx(a+bx^n)^{p-1} + b\int x^{m+n-1}dx(a+bx^n)^{p-1} \dots\dots (54).$$

By means of this equation, we may eliminate the last term of the equation (53); for if we multiply the equation (54) by $\frac{pn}{m}$, and add it to the equation (53), we shall find

$$\left(1+\frac{pn}{m}\right)\int x^{m-1}dx(a+bx^n)^p=(a+bx^n)^p\frac{x^m}{m}+\frac{pna}{m}\int x^{m-1}dx(a+bx^n)^{p-1},$$

and, multiplying by m, and dividing then by the constant factor on the first side, we shall obtain

$$\int x^{m-1}dx(a+bx^n)^p$$

$$=\frac{x^m}{(m+pn)}(a+bx^n)^p+\frac{pna}{m+pn}\int x^{m-1}dx(a+bx^n)^{p-1}\;\ldots\;(55).$$

By this formula, therefore, the integral of $x^{m-1}dx(a+bx^n)^p$ may be made to depend on another in which the index of the part within the brackets will be diminished by unity.

If now we put in this formula $p-1$ in place of p, the integral of $x^{m-1}dx(a+bx^n)^{p-1}$ will depend on that of $x^{m-1}dx(a+bx^n)^{p-2}$; by the same process, this, in its turn, will depend on that of $x^{m-1}dx(a+bx^n)^{p-3}$, and so on; so that the exponent of the part within the brackets will be successively p, $p-1$, $p-2$, $p-3$ . . . . . $p-n$ (by n representing the greatest integral number contained in p, which we suppose fractional).

If, then, we can obtain the integral of $x^{m-1}dx(a+bx^n)^{p-n}$, we shall have that in which the index of $a+bx$ is greater by unity, and so on, up to the integral of $x^{m-1}dx(a+bx^n)^p$, which we shall thus obtain in a finite number of algebraic terms.

327. If p were negative, the equation (55) would give

$$\int x^{m-1}dx(a+bx^n)^{p-1}=\frac{-x^m(a+bx^n)^p+(m+pn)\int x^{m-1}dx(a+bx^n)^p}{pna};$$

and making $p-1=p$, we should have

$$\int x^{m-1}dx(a+bx^n)^p$$

$$=\frac{-x^m(a+bx^n)^{p+1}+[m+(p+1)n]\int x^{m-1}dx(a+bx^n)^{p+1}}{(p+1)na}\;\ldots\;(56);$$

a formula in which, if we make p negative, the integral proposed will depend on another in which the index of the part within the brackets will become more nearly equal to zero by unity.

328. We may also diminish the index of x without the brackets. For this purpose, the first sides being equal, we shall equate to each other the second sides of the equations (53) and (54), which will give

$$\frac{x^m}{m}(a+bx^n)^p-\frac{pnb}{m}\int x^{m+n-1}(a+bx^n)^{p-1}dx$$

$$=a\int x^{m-1}dx(a+bx^n)^{p-1}+b\int x^{m+n-1}(a+bx^n)^{p-1}dx,$$

whence we shall deduce

$$\left(b+\frac{pnb}{m}\right)\int x^{m+n-1}(a+bx^n)^{p-1}dx=(a+bx^n)^p.\frac{x^m}{m}-a\int x^{m-1}dx(a+bx^n)^{p-1},$$

consequently

$$\int x^{m+n-1}(a+bx^n)^{p-1}dx=\frac{x^m(a+bx^n)^p-ma\int x^{m-1}dx(a+bx^n)^{p-1}}{b(m+pn)};$$

and making $m+n=m$, and $p-1=p$, this equation will become

$$\int x^{m-1}dx(a+bx^n)^p=\frac{x^{m-n}(a+bx^n)^{p+1}-(m-n)a\int x^{m-n-1}dx(a+bx^n)^p}{b(m+pn)}$$

. . . (57).

By means, then, of this formula, the integral will depend on another, in which the part x^{m-1}, without the brackets, will become x^{m-n-1}; this second integral will, in its turn, depend on a third, in which the part without the brackets will be x^{m-2n-1}; and continuing the process, the indices of x without the brackets will be successively $m-1$, $m-n-1$, $m-2n-1$, $m-3n-1$, ... $m-rn-1$; rn being the greatest multiple contained in m.

In the last of these operations, therefore, the index of x without the brackets, on the second side of the equation of reduction, will be $m-rn-1$; and consequently x, on the first side of that equation, will have for its index $m-(r-1)n-1$; thus making $m=m-(r-1)n$, in the formula (57), and representing the part integrated by X, that formula will give us

$$\int x^{m-(r-1)n-1}dx(a+bx^n)^p=\frac{X-(m-rn)a\int x^{m-rn-1}dx(a+bx^n)^p}{b[m-(r-1)n+pn]} \quad . \ (58).$$

If rn be equal to m, the coefficient $m-rn$ becomes 0 and consequently the part affected by the sign of integration, on the second side of the preceding equation, will vanish, and there will remain

$$\int x^{m-(r-1)n-1}dx(a+bx^n)^p=\frac{X}{bn(1+p)}.$$

This integral being determined accurately, all the others will be so likewise, and the proposed formula is therefore in this case integrable.

329. In the formula (57), which reduces the index of x without the brackets, m was supposed positive; to obtain the one which applies to the case in which m is negative, we derive from the formula (57),

$$\int x^{m-n-1}dx(a+bx^n)^p=\frac{x^{m-n}(a+bx^n)^{p+1}-b(m+pn)\int x^{m-1}dx(a+bx^n)^p}{(m-n)a},$$

whence, making $m-n=m$, we have

$$\int x^{m-1}dx(a+bx^n)^p = \frac{(a+bx^n)^{p+1}x^m - b(m+n+np)\int x^{m+n-1}dx(a+bx^n)^p}{ma}$$

$\ldots\ldots$ (59),

and by means of this formula, when the index of x without the brackets is negative, the integral will depend on another, in which the value of that index will be diminished by n; for the index of x, without the brackets, on the second side of the equation (59), being $m+n-1$, if we replace m by its negative value, which we will represent by m', that index will become $\ldots\ldots$ $-(m'-n)-1$, whilst that of x, without the brackets, on the first side, will be $-m'-1$; and considering only the numerical values of these indices, it is evident that $-(m'-n)-1$ will be greater than $-m'-1$ by n.

330. To give an application of these formulæ, let the expression be

$$\frac{x^m dx}{\sqrt{1-x^2}};$$

this being put under the form $x^m dx(1-x^2)^{-\frac{1}{2}}$, and compared with the one $x^{m-1}dx(a+bx^n)^p$, we shall have

$$m-1=m, \text{ or } m=m+1, \; a=1, \; b=-1, \; n=2, \; p=-\frac{1}{2};$$

and the index of the part within the brackets being less than unity, we must diminish the index without the brackets by substituting the above values in the formula (57), which will change it into

$$\int x^m dx(1-x^2)^{-\frac{1}{2}} = -x^{m-1}\frac{(1-x^2)^{\frac{1}{2}}}{m} + \frac{m-1}{m}\int x^{m-2}dx(1-x^2)^{-\frac{1}{2}},$$

or

$$\int \frac{x^m dx}{\sqrt{1-x^2}} = -\frac{x^{m-1}\sqrt{1-x^2}}{m} + \frac{m-1}{m}\int\frac{x^{m-2}dx}{\sqrt{1-x^2}} \quad \ldots\ldots (60).$$

If now we make successively

$$m=m-2, \text{ we have } \int\frac{x^{m-2}dx}{\sqrt{1-x^2}} = -x^{m-3}\frac{\sqrt{1-x^2}}{m-2} + \frac{m-3}{m-2}\int\frac{x^{m-4}dx,}{\sqrt{1-x^2}},$$

$$m=m-4 \ldots\ldots \int\frac{x^{m-4}dx}{\sqrt{1-x^2}} = -x^{m-5}\frac{\sqrt{1-x^2}}{m-4} + \frac{m-5}{m-4}\int\frac{x^{m-6}dx}{\sqrt{1-x^2}},$$

$$m=m-6 \ldots\ldots \int\frac{x^{m-6}dx}{\sqrt{1-x^2}}=-x^{m-7}\frac{\sqrt{1-x^2}}{m-6}+\frac{m-7}{m-6}\int\frac{x^{m-8}dx}{\sqrt{1-x^2}},$$

and so on.

The first of these equations will give us the value of $\int\frac{x^{m-2}dx}{\sqrt{1-x^2}}$, which being put in equation (60), we shall find

$$\int\frac{x^m dx}{\sqrt{1-x^2}}=-\sqrt{1-x^2}\left\{\frac{x^{m-1}}{m}+\frac{m-1}{m}\cdot\frac{x^{m-3}}{m-2}\right\}+\frac{m-1}{m}\cdot\frac{m-3}{m-2}\int\frac{x^{m-4}dx}{\sqrt{1-x^2}};$$

and substituting successively the values of

$$\int\frac{x^{m-4}dx}{\sqrt{1-x^2}},\ \int\frac{x^{m-6}dx}{\sqrt{1-x^2}},\ \&c.,$$

the last integral which we shall obtain, if m be even, will be

$$\int\frac{dx}{\sqrt{1-x^2}}=\sin^{-1}x;$$

but if m be odd, this last integral will be

$$\int\frac{xdx}{\sqrt{1-x^2}};$$

and since xdx is the differential of x^2, except as to the constant, we shall put $1-x^2=z$, which will give us

$$\int\frac{xdx}{\sqrt{1-x^2}}=\int-\frac{1}{2}\frac{dz}{\sqrt{z}}=\int-\frac{1}{2}z^{-\frac{1}{2}}dz=-z^{\frac{1}{2}}=-\sqrt{z}=-\sqrt{1-x^2};$$

the last integral being thus found, it follows that when m is an integral number the expression may always be integrated.

331. Let us take also for example

$$\frac{dx}{x^m\sqrt{1-x^2}};$$

this expression being written thus:

$$x^{-m}(1-x^2)^{-\frac{1}{2}}dx,$$

and compared with the formula (59), in order to diminish the index without the brackets, we shall have

$$m-1=-m,\ a=1,\ b=-1,\ n=2,\ p=-\frac{1}{2},$$

by means of which values the formula (59) will become

$$\int x^{-m}dx(1-x^2)^{-\frac{1}{2}}=\frac{(1-x^2)^{\frac{1}{2}}}{1-m}x^{1-m}+\frac{2-m}{1-m}\int x^{-m+2}dx(1-x^2)^{-\frac{1}{2}},$$

or

$$\int\frac{dx}{x^m\sqrt{1-x^2}}=-\frac{\sqrt{1-x^2}}{(m-1)x^{m-1}}+\frac{m-2}{m-1}\int\frac{dx}{x^{m-2}\sqrt{1-x^2}}\ \ldots\ (61).$$

and if m be an even number, for example 8, the integral of $\frac{dx}{x^8\sqrt{1-x^2}}$ will depend on that of $\frac{dx}{x^6\sqrt{1-x^2}}$; this, by virtue of the same formula, will depend on the integral of $\frac{dx}{x^4\sqrt{1-x^2}}$, and we shall come at length to $m=2$; in which last case the formula (61) will give

$$\int\frac{dx}{x^2\sqrt{1-x^2}}=-\frac{\sqrt{1-x^2}}{x}+C;$$

so that, by these successive substitutions, we obtain the integral when m is even.

In the case in which m is odd, for example 7, the values 7, 5, 3, being successively substituted for m in the formula (61), we cannot proceed to $m=1$; for, on this hypothesis, the coefficient $\frac{m-2}{m-1}$ of the second integral will become $-\frac{1}{0}=-\infty$; the least value, therefore, that can be given to m will be $m=3$; and on this hypothesis the formula (61) will become

$$\int\frac{dx}{x^3\sqrt{1-x^2}}=-\frac{\sqrt{1-x^2}}{2x^2}+\frac{1}{2}\int\frac{dx}{x\sqrt{1-x^2}}.$$

To integrate the expression

$$\frac{dx}{x\sqrt{1-x^2}},$$

we shall put $x=\frac{1}{z}$, which will give us

$$dx=-\frac{dz}{z^2},\quad \sqrt{1-x^2}=\frac{\sqrt{z^2-1}}{z},$$

and consequently

$$\frac{dx}{x\sqrt{1-x^2}}=-\frac{dz}{\sqrt{z^2-1}}.$$

But we have found, art. 288

$$\int \frac{dx}{\sqrt{x^2-1}} = \log\left(x+\sqrt{x^2-1}\right),$$

and, therefore, changing x into z, we shall have

$$\int -\frac{dz}{\sqrt{z^2-1}} = -\log\left(z+\sqrt{z^2-1}\right),$$

and putting for z its value $\frac{1}{x}$, we obtain

$$\int -\frac{dx}{x\sqrt{1-x^2}} = -\log\left(\frac{1}{x}+\sqrt{\frac{1}{x^2}-1}\right)+C$$

$$= -\log\left(\frac{1+\sqrt{1-x^2}}{x}\right)+C.$$

Thus the formula $\frac{dx}{x^m\sqrt{1-x^2}}$ may be integrated, whether we take m even or odd.

Integration of functions of sines and cosines.

332. The integration of quantities involving sines and cosines depending on the possibility of developing $\cos^2 x$, $\cos^3 x$, $\cos^4 x$, &c. in functions of the expressions $\cos x$, $\cos 2x$, $\cos 3x$, &c.; we will proceed first to show how this may be accomplished by trigonometry alone (*note fifth*).

If, in the formula

$$\cos(a+b) = \cos a \,.\, \cos b - \sin a \,.\, \sin b \quad . \quad . \quad . \quad (62),$$

we make $a=b$, we shall have

$$\cos 2a = \cos^2 a - \sin^2 a = \cos^2 a - (1-\cos^2 a)$$
$$= 2\cos^2 a - 1;$$

whence we derive

$$\cos^2 a = \tfrac{1}{2} + \tfrac{1}{2}\cos 2a,$$

and multiplying this equation by $\cos a$, it becomes

$$\cos^3 a = \tfrac{1}{2}\cos a + \tfrac{1}{2}\cos a \quad \cos 2a \quad . \quad . \quad (63)$$

But if to the equation (62) we add the one,

$$\cos(b-a)=\cos a \,.\, \cos b+\sin a \,.\, \sin b,$$

we shall obtain

$$\cos a \,.\, \cos b=\tfrac{1}{2}\cos(a+b)+\tfrac{1}{2}\cos(b-a)\ ,$$

and making $b=2a$, we shall have

$$\cos a \,.\, \cos 2a=\tfrac{1}{2}\cos 3a+\tfrac{1}{2}\cos a\ ;$$

eliminating, therefore, $\cos a . \cos 2a$ betwixt this equation and the equation (63), we shall find

$$\cos^3 a=\tfrac{3}{4}\cos a+\tfrac{1}{4}\cos 3a\ ;$$

and by the same process may be calculated the higher powers of $\cos a$.

333. This being premised, when we have to integrate the expression $\cos^m x dx$, in which m is an integral number, we must put for $\cos^m x$ its development, which, according to what has preceded, will contain only terms of the series

$$constant,\ \cos x,\ \cos 2x,\ \cos 3x,\ \cos 4x,\ \&c.\ ;$$

so that the whole will be reduced to the knowing how to integrate the expression $\cos mxdx$.

For this purpose, we must observe, that if, in the equation

$$d \sin z=\cos z dz,$$

we make $z=mx$, we shall have

$$d \sin mx=\cos mx . mdx\ ;$$

and therefore

$$\int \cos mxdx=\frac{\sin mx}{m}\ ;$$

and, similarly, we should find that

$$\int \sin mx dx = -\frac{\cos mx}{m}.$$

Taking, for example, $\cos^2 x dx$, and putting for $\cos^2 x$ its value $\frac{1}{2}+\frac{1}{2}\cos 2x$, we shall have

$$\int \cos^2 x dx = \int (\tfrac{1}{2}+\tfrac{1}{2}\cos 2x) dx = \frac{x}{2} + \tfrac{1}{4}\sin 2x + \mathrm{C}.$$

334. If we wished to integrate $\sin^m x dx$, we might proceed in a similar way; or, otherwise, representing the complement of x by z, we should have

$$x = \tfrac{1}{2}\pi - z,\ dx = -dz,\ \sin x = \cos z,$$

which would therefore change the formula $\sin^m x dx$ into the one $-\cos^m z dz$, and we might integrate as above.

335. Taking the most general case $\sin^m x \cos^n x dx$; if m be even, we will put $m=2m'$, when we shall have to integrate

$$\sin^{2m'} x \cos^n x dx = (1-\cos^2 x)^{m'} \cos^n x dx;$$

and developing $(1-\cos^2 x)^{m'}$, and multiplying by $\cos^n x dx$, we shall obtain a series of terms, each of the form $\cos^k x dx$, which we shall integrate as above.

If m be odd, we must put $m=2m'+1$, when we shall have

$$\begin{aligned}\sin^m x \cos^n x dx &= \sin^{2m'} x \cos^n x \sin x dx \\ &= (1-\cos^2 x)^{m'} \cos^n x \times -d\cos x;\end{aligned}$$

making $\cos x = z$, we shall change this expression into

$$-(1-z^2)^{m'} z^n dz;$$

and m' and n being, by hypothesis, integers, we may develop and integrate.

336. Applying this process to the expressions

$$\frac{\cos^m x dx}{\sin^n x},\quad \frac{\sin^n x dx}{\cos^m x},$$

since the second comes under the form of the other, by making

$x = \frac{\pi}{2} - z$, we shall consider only the first: and if m be even, we shall assume $m = 2m'$, when we shall have

$$\frac{\cos^m x dx}{\sin^n x} = \frac{(1 - \sin^2 x)^{m'} dx}{\sin^n x}$$

$$= \frac{1 - m'\sin^2 x + m'\frac{m'-1}{2}\sin^4 x^2 + \&c.}{\sin^n x} dx,$$

an expression, the integral of which will depend on those of $\sin^l x dx$ and $\frac{dx}{\sin^k x}$.

If m be odd, making $m = 2m' + 1$, we shall have

$$\frac{\cos^m x dx}{\sin^n x} = \frac{(1 - \sin^2 x)^{m'} \cos x dx}{\sin^n x} = (1 - m'\sin^2 x + \&c.)\frac{\cos x dx}{\sin^n x},$$

an expression, the integral of which will depend on those of $\sin^l x \cos x dx$ and $\frac{dx \cos x}{\sin^k x}$.

The integrals of $\sin^l x dx$ and $\sin^l x \cos x dx$ have already been treated of; to integrate $\frac{dx \cos x}{\sin^k x}$ we must put $\sin x = z$, whence $dx \cos x = dz$, and consequently

$$\int \frac{dx \cos x}{\sin^k x} = \int \frac{dz}{z^k} = \int z^{-k} dz = \frac{z^{-k+1}}{1-k} + C.$$

In respect to the integral of $\frac{dx}{\sin^k x}$, the same transformation will change this expression into $\frac{dz}{z^k (1 - z^2)^{\frac{1}{2}}}$, a formula which we know already how to integrate.

337. If, lastly, we have to integrate $\frac{dx}{\cos^m x \sin^n x}$ we must multiply the expression by $\cos^2 x + \sin^2 x$, a quantity equivalent to unity, when we shall have

$$\frac{dx}{\cos^m x \sin^n x} = \frac{dx}{\cos^{m-2} x \sin^n x} + \frac{dx}{\cos^m x \sin^{n-2} x};$$

by which the sum of the indices of the denominator will be diminished; and repeating the operation, and setting apart successively all the fractions, which, in their denominators, contain powers of the sine alone, or the cosine alone (since we know how to integrate these fractions from what has preceded), at the last operation we shall meet with terms still containing powers of the sine and cosine, or which will be of the following forms:

$$\frac{dx}{\cos x \sin x},\ \frac{dx}{\cos x},\ \frac{dx}{\sin x}.$$

To integrate $\frac{dx}{\cos x \sin x}$, we must multiply the numerator by $\cos^2 x + \sin^2 x$, and we shall have

$$\frac{dx}{\cos x \sin x} = dx . \frac{\cos x}{\sin x} + dx \frac{\sin x}{\cos x} = \frac{d.\sin x}{\sin x} - \frac{d.\cos x}{\cos x},$$

the integral of which is

$$\log \sin x - \log \cos x + \log C = \log C \operatorname{tang} x.$$

To integrate $\frac{dx}{\sin x}$, we must put $\cos x = z$, and we shall have

$$dx = -\frac{dz}{\sin x},\quad \frac{dx}{\sin x} = -\frac{dz}{\sin^2 x} = -\frac{dz}{1-z^2},$$

an expression integrable by the method of rational fractions.

In regard to $\frac{dx}{\cos x}$, we shall suppose $\sin x = z$, and we shall find

$$\int \frac{dx}{\cos x} = \int \frac{dz}{\sqrt{1-z^2}}.$$

338. In general, we may always transform expressions con-

taining sines and cosines into others which do not contain them, by simply equating $\sin x$ or $\cos x$ to a new variable z.

For example, if in the expression $\sin^m x \cos^n x dx$, we suppose $\sin x = z$, we shall have

$$\cos x = \sqrt{1-z^2},\ dx = \frac{dz}{\sqrt{1-z^2}},$$

and substituting, we shall find

$$\sin^m x \cos^n x dx = z^m (1-z^2)^{\frac{n}{2}} (1-z^2)^{-\frac{1}{2}} dz = z^m (1-z^2)^{\frac{n-1}{2}} dz,$$

an expression which comes under the form of binomial differentials.

The method of integration by parts may also be applied immediately to the expression* $\sin^m x \cos^n x dx$.

339. Lastly, trigonometrical formulæ also may in some cases be employed with advantage. To integrate, for example, $\sin mx \cos nx dx$; since Trigonometry gives us

$$\sin a \,.\cos b = \tfrac{1}{2}\sin(a+b) + \tfrac{1}{2}\sin(a-b),$$

by comparing the expressions $\sin mx \cos nx dx$ with this formula, we shall find

$$\sin mx \cos nx dx = \tfrac{1}{2}\sin[(m+n)x]\,dx + \tfrac{1}{2}\sin[(m-n)x]\,dx,$$

and the integral will be, art. 333,

$$\mathrm{C} - \tfrac{1}{2}\frac{\cos[(m+n)x]}{m+n} - \tfrac{1}{2}\,.\frac{\cos[(m-n)x]}{m-n}.$$

On the integration of exponential and logarithmic quantities.

340. It has been demonstrated, art. 37, that, taking the

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* To compare the expression with udv, we must decompose it thus:
$\sin^{m-1}x \cos^n x \sin x dx = -\sin^{m-1}x\, d.\dfrac{\cos^{n+1}x}{n+1}$.

logarithms in the Napierian system, we have $da^x = a^x dx \log a$, and therefore, reciprocally,

$$\int a^x dx = \frac{a^x}{\log a},$$

a form which will serve to integrate the general expression $a^x X dx$, in which X is a function of x. For this purpose we must write the expression thus: $X . a^x dx$; and integrating by the method of parts we shall have

$$\int X . a^x dx = \frac{X . a^x}{\log a} - \int \frac{a^x}{\log a} dX \;\ldots\ldots\; (64),$$

the function X and its derivatives being then differentiated successively, we shall deduce $dX = X'dx$, $dX' = X''dx$, &c.; and therefore

$$\int \frac{a^x}{\log a} dX \text{ or } \int \frac{X'}{\log a} a^x dx = \frac{X'}{(\log a)^2} a^x - \int \frac{a^x}{(\log a)^2} dX';$$

whence, substituting this value in the place of the last term in the equation (64), we shall obtain

$$\int X a^x dx = \frac{X . a^x}{\log a} - \frac{X' . a^x}{(\log a)^2} + \int \frac{a^x}{(\log a)^2} dX'.$$

This operation being thus continued, we shall arrive at length at the development

$$\int X a^x dx = a^x \left(\frac{X}{\log a} - \frac{X'}{(\log a)^2} + \frac{X''}{(\log a)^3} - \frac{X'''}{(\log a)^4} \cdots \pm \frac{X^{(n)}}{(\log a)^{n+1}} \right.$$
$$\mp \int \frac{a^x dX^{(n)}}{(\log a)^{n+1}};$$

and if, taking the series of the differential coefficients ...... X', X'', X''' .... $X^{(n)}$, the last of these coefficients be constant, we shall have $dX^{(n)} = 0$, and therefore the part under the integral sign will vanish.

341. Let us take, for example, $X = x^3$; we deduce thence

$$X' = 3x^2,\ X'' = 2.3.x,\ X''' \text{ or } X^{(n)} = 3.2;$$

and therefore,

$$\int x^3 a^x dx = a^x\left(\frac{x^3}{\log a} - \frac{3x^2}{(\log a)^2} + \frac{2.3x}{(\log a)^3} - \frac{2.3}{(\log a)^4}\right).$$

If we make a equal to the number e, which is the base of the Napierian system, $\log a$ becomes $\log e$, and since $\log e = 1$, by virtue of the equation $e = e^{\log e}$, the preceding series will become

$$\int x^3 e^x dx = e^x(x^3 - 3x^2 + 2.3x - 2.3).$$

342. We may arrive also at another development of $\int a^x X dx$ in the following manner: making . . . . $\int X dx = P, \int P dx = Q, \int Q dx = R$, &c., and integrating by the method of parts, we shall have

$$\int a^x X dx = a^x P - \int a^x \log a\, P dx \ \ldots\ (65),$$

$$\int a^x \log a . P dx = a^x \log a . Q - \int a^x (\log a)^2 Q dx;$$

substituting in the equation (65) it will become

$$\int a^x . X dx = a^x P - a^x \log a . Q + \int a^x (\log a)^2 Q dx;$$

and continuing to integrate by parts, we shall have generally

$$\int a^x X dx = a^x [P - Q \log a + R (\log a)^2 - \&c.] \pm \int Z a^x (\log a)^n dx.$$

343. If we apply this formula to the case in which $X = \frac{1}{x^5}$, we shall find

$$P = -\frac{1}{4x^4},\quad Q = \frac{1}{3.4x^3},\quad R = -\frac{1}{2.3.4x^2},\quad Z = \frac{1}{2.3.4x}$$

and therefore

$$\int \frac{a^x dx}{x^5} = a^x\left(-\frac{1}{4x^4} - \frac{\log a}{3.4x^3} - \frac{(\log a)^2}{2.3.4x^2}\right) - \frac{(\log a)^3}{2.3.4}\int \frac{a^x dx}{x}.$$

The integral of $\frac{a^x dx}{x}$ is a transcendental function, the exact value of which has never yet been determined.

344. We see, generally, that whatever negative and integral value we give to the exponent of x, we shall always come at last to the transcendental $\int \frac{a^x dx}{x}$; for the exponents of x in the functions P, Q, R, &c. being successively diminished by unity, the last of these functions must be of the form $\frac{A}{x}$, and consequently the last integral will be

$$\int \frac{Aa^x}{x} dx = A\int \frac{a^x dx}{x},$$

since A is constant.

To obtain an approximate value of the integral of $\frac{Aa^x dx}{x}$, we have no other means than to substitute in the expression the development of a^x, which, as we have seen, is

$$1 + x \log a + \frac{x^2}{2}(\log a)^2 + \frac{x^3}{2.3}(\log a)^3 + \&c.,$$

and then to integrate each term separately.

345. If in the equation $\frac{du}{u} = d.\log u$, or $du = ud \log u$, we make $u = x^y$, we shall have

$$dx^y = x^y d \log x^y;$$

thus, whenever we can decompose a differential into two parts, one of which may be represented by x^y, and the other by $d. \log x^y$, the integral will be $x^y + C$.

346. The integration by parts may be applied also to the expression $Xdx(\log x)^n$; for if we represent the integral of Xdx by $X_,$, we shall have

$$\int Xdx\,(\log x)^n = X_,(\log x)^n - n\int \frac{X_,}{x} dx (\log x)^{n-1};$$

and this last integral may be made in its turn to depend on another of the form $X_{,,}dx\,(\log x)^{n-2}$, and so on.

Bernouilli's series.

347. We have seen that differential expressions are frequently not integrable until they have been reduced into the form of a series, and that, for this purpose, representing by Xdx a differential in which X is any function whatever of x, we have first to reduce the function represented by X into the form of a series, and then to integrate, after having substituted the development thus obtained in the formula Xdx.

The series of Bernouilli has the advantage of reducing $\int Xdx$ into a series, even before we have given the form of X, and is in the integral calculus what Taylor's series is in the differential: it is proved in the following manner:

Proceeding first to integrate Xdx by the method of parts, we shall compare $\int Xdx$ with the first term of the formula

$$\int udv = uv - \int vdu,$$

when we shall have

$$X = u,\quad dx = dv;$$

the integration by parts will therefore be effected by making

$$\int Xdx = Xx - \int xdX \; \ldots . \; (66);$$

and the integral being taken in respect to x, we have

$$dX = \frac{dX}{dx}dx,$$

and consequently

$$\int xdX = \int \frac{dX}{dx,} xdx.$$

Integrating again by parts, u will be represented, in this case, by $\frac{dX}{dx}$, and dv by xdx, so that we shall have $v = \frac{x^2}{2}$ and we shall find

$$\int\frac{dX}{dx}\,.\,xdx=\frac{dX}{dx}\,.\,\frac{x^2}{2}-\int\frac{x^2}{2}\,\frac{d^2X}{dx}\,;$$

or, putting the fraction $\frac{1}{2}$ before the sign of integration,

$$\int\frac{dX}{dx}xdx=\frac{dX}{dx}\,.\,\frac{x^2}{2}-\tfrac{1}{2}\int x^2.\frac{d^2X}{dx}\;\ldots\;(67\,;$$

and replacing $\frac{d^2X}{dx}$ by $\frac{d^2X}{dx^2}dx$, and replacing the process, we shall obtain

$$\int x^2.\frac{d^2X}{dx}\text{ or }\int\frac{d^2X}{dx^2}.x^2dx=\tfrac{1}{3}x^3\frac{d^2X}{dx^2}-\tfrac{1}{3}\int x^3\,.\,\frac{d^3X}{dx^2}\;\ldots\ldots\;(68)\,;$$

whence, substituting in the equation (66) the value of the first side of the equation (67), and in the result substituting the value of the first side of the equation (68), and so on, we shall obtain

$$\int Xdx=Xx-\frac{dX}{dx}\,.\,\frac{x^2}{1.2}+\frac{d^2X}{dx^2}\,.\,\frac{x^3}{1.2.3}-\&c.+\textit{constant}.$$

On the quadrature of curves.

348. Let s be the area ABMP (fig. 51) of a plane curve; Fig. 51. if the abscissa AP $=x$ become AP$'=x+h$, the area s will become

$$\textit{area}\ \text{ABM}'\text{P}'=s+\frac{ds}{dx}h+\frac{d^2s}{dx^2}\,\frac{h^2}{1.2}+\&c.\,;$$

and we shall have therefore

$$\textit{curvilinear area}\ \text{PMM}'\text{P}'=\frac{ds}{dx}\,h+\frac{d^2s}{dx^2}\,\frac{h^2}{2}+\&c.$$

Now this area is comprised between the two rectangles PM′, and P′M, for which we easily obtain the analytical expressions

$$\textit{rectangle}\ \mathrm{PM'} = \mathrm{P'M'} \times \mathrm{PP'} = f(x+h)h,$$
$$\textit{rectangle}\ \mathrm{P'M} = \mathrm{PM} \times \mathrm{PP'} = fx \,.\, h\,;$$

the ratio of these rectangles therefore is

$$\frac{f(x+h)h}{fx \,.\, h} = \frac{f(x+h)}{fx},$$

and, in the case of the limit, this ratio is reduced to

$$\frac{fx}{fx} = 1.$$

But the curvilinear surface PMM'P', being comprised between the two rectangles, must differ less from the rectangle P'M than the rectangle PM' does; and consequently, if, in the case of the limit, we have $\frac{\mathrm{PM'}}{\mathrm{P'M}} = 1$, much more will unity be the limit of the ratio

$$\frac{\textit{area}\ \mathrm{PMM'P'}}{\textit{rectangle}\ \mathrm{P'M}}.$$

Replacing therefore the terms of this ratio by their analytical expressions, we shall have

$$\frac{\frac{ds}{dx}h + \frac{d^2s}{dx^2}\frac{h^2}{2} + \&c.}{fx \,.\, h} = \frac{\frac{ds}{dx} + \frac{d^2s}{dx^2}\frac{h}{2} + \&c.}{fx}$$

and, passing to the limit, by making $h = 0$, we shall find

$$\frac{ds}{dxfx} = 1\,;$$

whence $ds = fx.dx$; and putting for fx its value, we shall have

$$ds = ydx \ldots . (69).$$

Fig. 59. 349. We might also determine the differential of the area of a curve, by the method of infinitesimals, in the manner following (fig. 59):

$$\textit{trapezium}\ \mathrm{PMM'P'} = \frac{\mathrm{PM}+\mathrm{P'M'}}{2}\times \mathrm{PP'}$$
$$= \frac{y+(y+dy)}{2}dx = ydx + \frac{dxdy}{2};$$

and rejecting $dxdy$ as an infinitesimal of the second order, there will remain ydx for the differential.

350. As a first application, we will determine the area of the portion BMP (fig. 52) of a parabola. Fig. 52.

Let $y^2 = mx$ be the equation of the parabola, and B the origin; we find then, by differentiating, $2ydy = mdx$; therefore $dx = \frac{2y}{m}dy$, and consequently $ydx = \frac{2y^2}{m}dy$; whence, integrating, we have

$$\int \frac{2y^2}{m}dy = \frac{2}{3}\frac{y^3}{m} + C \quad . \; . \; . \; (70).$$

To determine the constant, we must observe that when $y=0$, the integral which expresses the area sought is also 0; this hypothesis, therefore, reduces the equation (70) to $0=0+C$, and

$$\int ydx = \frac{2}{3}\frac{y^3}{m} = \frac{2}{3}\frac{y}{m}\cdot y^2 = \frac{2}{3}\frac{y}{m}\cdot mx = \frac{2}{3}xy.$$

351. We have now some important observations to make respecting the determination of the constant; and for this purpose we shall solve the same problem, taking the parabola whose equation is

$$y^2 = m + nx \quad . \; . \; . \; . \; (71).$$

In this case the origin of the abscissæ is no longer at the vertex of the curve; for on making $y=0$, the equation (71) gives $x = -\frac{m}{n}$; and since this abscissa must terminate at the point

Fig. 53. B, in which $y=0$ (fig. 53), if we draw from B the line . . . . $\mathrm{BA}=\frac{m}{n}$, A will be the origin.

This being premised, on proceeding as before, we shall find

$$2ydy=ndx,\ ydx=\frac{2y^2}{n}dy,\ \text{and} \int ydx=\frac{2}{3}\cdot\frac{y^3}{n}+\mathrm{C}\ ..\ (72),$$

and to determine the constant, we must observe that the area ADMP, which here represents the integral, must become 0 when the ordinate MP coincides with AD.

Now AD being the ordinate which passes through the origin A where the abscissa $x=0$, the equation (71) will give us, on this hypothesis,

y or $\mathrm{AD}=\sqrt{m}$; and making, therefore, $\int ydx=0$, and $y=\sqrt{m}$, these values will reduce the equation (72) to $0=\frac{2m^{\frac{3}{2}}}{3n}+\mathrm{C}$;

whence we deduce $\mathrm{C}=-\frac{2}{3}\frac{m^{\frac{3}{2}}}{n}$, and consequently the integral sought is

$$\int ydx=\frac{2}{3}\frac{y^3}{n}-\frac{2m^{\frac{3}{2}}}{3n}=area\ \mathrm{ADMP}.$$

352. In what has preceded, we have deduced from the equation of the curve the value of dx, in order to substitute it in the formula ydx, and then integrate. We might proceed otherwise, putting in that expression the value of y instead of that of dx; for to obtain the integral, it is sufficient that the differential proposed contain only one variable; thus, in making the substitution, we may choose the one which requires the least calculation.

353. An integral, such as $\int fxdx$, may always represent the area of a curve, the equation to which is $y=fx$; for this equation being given, if we substitute the value of y in the for-

mula $\int ydx$, we shall have $\int fxdx$ for the area of that curve. It is on this account that when a problem conducts us to the integrating a function of only one variable, the problem is said to be reduced to quadratures.

354. Let X be a function of x, and suppose that by integrating Xdx we have obtained

$$\int Xdx = Fx + C;$$

this integral, in which the constant C is not yet determined, bears the name of the *general indefinite integral*, or, more simply, of the *indefinite integral*, and it is complete when it contains the arbitrary constant C.

355. If, by any hypothesis, we determine this constant C; if, for instance, we suppose that $\int Xdx$ ought to vanish when $x=a$, the equation (73) gives, in this case, $0=Fa+C$, whence $C=-Fa$, and the equation (73) becomes

$$\int Xdx = Fx - Fa;$$

this integral $Fx-Fa$ is then a particular integral, and we see that a differential expression has an indefinite number of particular integrals, since we may make an infinite number of hypotheses respecting the constant.

356. In making the hypothesis of the integral: being 0 when $x=a$, we suppose that taking an abscissa $AB=a$ (fig. 54), the Fig. 54. surface is comprised betwixt the limit BD and the indefinite limit MP, which corresponds to $AP=x$; the operation, therefore, by which we determine a particular integral is the same with that which would fix the position of the limit BD, from which we reckon the integral. The second limit MP will, in its turn, be fixed invariably, if we give to x a determinate value b; and then the particular integral $\int ydx = Fx - Fa$ will become

$$\int ydx = Fb - Fa, \quad . \; . \; . \; . \quad (74),$$

and the surface BDMP will be no longer arbitrary. In this case, the integral bears the name of *definite integral,* and is said to be taken from $x=a$ to $x=b$.

357. We will now investigate the definite integral of $x^m dx$, presuming, of course, that we have given the two values a and b, which satisfy the indefinite integral

$$\frac{x^{m+1}}{m+1}+C \quad . \; . \; . \quad (75).$$

Suppose that the first corresponds to $\int y dx=0$; we shall have, then,

$$\frac{a^{m+1}}{m+1}+C=0,$$

and the particular integral will be

$$\int x^m dx=\frac{x^{m+1}}{m+1}-\frac{a^{m+1}}{m+1}.$$

Put now $x=b$, and we shall have for the definite integral

$$\int x^m dx=\frac{b^{m+1}}{m+1}-\frac{a^{m+1}}{m+1}.$$

358. We might arrive also at this integral by making successively $x=a$, and $x=b$ in the indefinite integral, and subtracting the first result

$$\frac{a^{m+1}}{m+1}+C$$

from the second

$$\frac{b^{m+1}}{m+1}+C;$$

observing only that, in taking this difference, the part subtracted must be the value of the function of x at the origin of the integral.

359. As a third application, we will determine the area of a right-angled triangle ABC (fig. 55): in this case the equation of the straight line AC being $y=ax$, on putting this value of y in the formula ydx, we obtain $axdx$, whence

$$\int ydx=\int axdx=\frac{ax^2}{2}+C;$$

and the area being 0 when $x=0$, the constant is equal to 0, and therefore

$$area\ ABC=\frac{ax^2}{2}=\frac{x}{2}\times ax=\frac{xy}{2}.$$

360. If in the formula ydx we put the value of y, deduced from the equation of the circle, we shall find $\int dx\sqrt{a^2-x^2}$ for the expression of the area of the circle; and we saw (art. 282) that this integral had for its value

$$\frac{x}{2}\sqrt{a^2-x^2}+\tfrac{1}{2}a^2\sin^{-1}\frac{x}{a}+C.$$

The part $\frac{1}{2}a^2\sin^{-1}\frac{x}{a}$ cannot be determined except by supposing that the ratio of the diameter to the circumference * is known; and we see, therefore, that the integration of $dx\sqrt{a^2-x^2}$ cannot lead to the solution of the problem of the quadrature of the circle; which is likewise the case with the quadrature of the ellipse, which depends on

$$\frac{b}{a}\int dx\sqrt{a^2-x^2}.$$

If we compare these two expressions, we shall derive the proportion

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* If, for example, $x=\frac{1}{6}a$, we have $\frac{x}{a}=\frac{1}{6}$, and we must proceed as in art. 278, to determine the corresponding arc.

$$\textit{area of ellipse} : \textit{area of circle} :: \frac{b}{a}\int dx\sqrt{a^2-x^2} : \int dx\sqrt{a^2-x^2},$$

or,

$$\textit{area of ellipse} : \textit{area of circle} :: \frac{b}{a} : 1\ ;$$

whence we have

$$\textit{area of ellipse} = \frac{b}{a}, \textit{ area of circle} = \frac{b}{a}\pi a^2 = \pi ab.$$

On the rectification of curves.

361. To rectify a curve is to obtain a straight line equal to an arc of the curve. Now we found, art. 159, that the differential of an arc of a curve had for its expression

$$ds = \sqrt{dx^2+dy^2} \quad . \; . \; . \quad (76)\ ;$$

if, therefore, an equation be given between two variables, x and y, and we wish to rectify the curve to which it belongs, we must differentiate the equation, and substitute the value of dx or dy, thus determined, in the expression (76) ; the quantity under the root will then involve only one variable ; and if we can obtain the integral, the curve is rectifiable.

362. Let us take, for example, the curve[*] found art. 165, the equation to which is $y^3 = nx^2$; this equation, being differentiated, gives us

$$3y^2dy = 2nxdx\ ;$$

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[*] It bears the name of the *semi-cubical parabola.* This equation, as well as that of the common parabola, is only a particular case of the general equation $y^m = ax^n$, which, for that reason, is called the *equation of the parabola of all orders.*

We may also consider the equation $xy = a$ of the hyperbola between the asymptotes, as a particular case of the equation $x^m y^n = a^{m+n}$, which is therefore called the *equation of the hyperbola of all orders.*

whence we deduce

$$dx=\frac{3y^2dy}{2nx},\ dx^2=\frac{9}{4}\,\frac{y^4}{n^2x^2}dy^2=\frac{9}{4}\,\frac{y^4}{ny^3}dy^2=\frac{9}{4}\,\frac{y}{n}dy^2$$

and substituting, we have

$$\sqrt{dx^2+dy^2}=\sqrt{\left(\frac{9}{4}\,\frac{y}{n}+1\right)dy^2}=dy\sqrt{\frac{9}{4}.\frac{y}{n}+1}.$$

To integrate this, since dy is the differential of the expression under the root, except as to the constant, we shall put, art. 271,

$$\frac{9}{4}\,\frac{y}{n}+1=z\ ;$$

whence we deduce

$$dy=\frac{4n}{9}dz\ ;$$

and substituting, we shall have

$$\sqrt{dx^2+dy^2}=\frac{4n}{9}z^{\frac{1}{2}}dz\ ;$$

and therefore

$$\int\sqrt{dx^2+dy^2}=\frac{4n}{9}\,\frac{z^{\frac{3}{2}}}{\frac{3}{2}}=\frac{8n}{27}z^{\frac{3}{2}}+C,$$

or, replacing the value of z,

$$\int\sqrt{dx^2+dy^2}=\frac{8n}{27}\left(\frac{9}{4}\,\frac{y}{n}+1\right)^{\frac{3}{2}}+C.$$

To determine the constant, we see, from the nature of the equation of the curve, that, at the origin of the abscissæ, y is 0 ; whence, supposing that the integral also is 0 at that point, we have

$$0=\frac{8n}{27}+C,\ C=-\frac{8n}{27}\ ;$$

and consequently

$$\int\sqrt{dx^2+dy^2}=\frac{8n}{27}\sqrt{\left[\frac{9}{4}\left(\frac{x}{n}\right)^{\frac{2}{3}}+1\right]^3}-\frac{8n}{27}.$$

If $x=a$, the arc s comprised between the limits $x=0$, and $x=a$, will be

$$s=\frac{8n}{27}\sqrt{\left[\frac{9}{4}\left(\frac{a}{n}\right)^{\frac{2}{3}}+1\right]^3}-\frac{8n}{27}.$$

363. The equation of the cycloid, art. 200, gives $dx^2=\frac{y^2dy^2}{2ay-y^2}$; this value, therefore, being substituted in the formula (76), we shall find

$$ds=\sqrt{dy^2+\frac{y^2dy^2}{2ay-y^2}}=dy\sqrt{\frac{2ay}{2ay-y^2}}=dy\sqrt{\frac{2a}{2a-y}}$$
$$=(2a)^{\frac{1}{2}}\times\frac{dy}{(2a-y)^{\frac{1}{2}}};$$

and since $-dy$ expresses the differential of the part under the root, we shall put (art. 271) $2a-y=z$, when we shall have

$$dy\sqrt{\frac{2a}{2a-y}}=-(2a)^{\frac{1}{2}}z^{-\frac{1}{2}}dz,$$

an equation which, being integrated, gives

$$\int dy\sqrt{\frac{2a}{2a-y}}=-(2a)^{\frac{1}{2}}.2z^{\frac{1}{2}}+C=-2\sqrt{2az}+C;$$

or, restoring the value of y,

$$\int dy\sqrt{\frac{2a}{2a-y}}=-2\sqrt{2a(2a-y)}+C\;.\;.\;.\;.\;(77).$$

To determine the constant, we will take the integral so that it shall vanish when $y=2a$; on this hypothesis, the equation (77) is reduced to $0=0+C$, which shows that there is no constant to be added, and the arc of the cycloid will extend from the point B (fig. 57), where $y=2a$, to the point M, whose coordinates are x and y. The absolute value of the arc MB being $2\sqrt{2a(2a-y)}$, it will be observed that $\text{BE}=2a-y$, and therefore

Fig. 57.

$$2\sqrt{2a(2a-y)}=2\sqrt{BD\times BE}=2BG;$$

whence it follows, that the arc MB of the cycloid is double of the chord BG, consequently arc AB$=$2BD.

On the determination of the surface of a solid of revolution.

364. If a curve BC (fig. 51), lying in one plane, revolve about the axis AX, it will generate a solid of revolution. We will now investigate the expression for the differential of the surface of the solid which is thus generated.

For this purpose, let AP $= x$, PM $= y$, PP$'$ $= h$, and consequently

$$PM=fx=y$$

$$P'M'=f(x+h)=y+\frac{dy}{dx}h+\frac{d^2y}{dx^2}\frac{h^2}{1.2}+\&c.:$$

then the ordinates MP and M$'$P$'$ describing, in the course of their revolution, unequal circles, these circles will be the bases of a truncated cone, of which the chord MM$'$ will be the side; and the expression for the surface of this truncated cone will be

$$\frac{circ.\,PM+circ.\,P'M'}{2}\times chord\,MM',$$

or, representing by $1:\pi$ the ratio of the diameter to the circumference,

$$\frac{2\pi.PM+2\pi P'M'}{2}\times chord\,MM'=\pi(PM+P'M')chord\,MM';$$

whence, putting for the ordinates PM, P$'$M$'$, their analytical values, we shall have

$$surface\ of\ truncated\ cone\ MM'$$

$$=\pi\left(2y+\frac{dy}{dx}h+\frac{d^2y}{dx^2}\frac{h^2}{1.2}+\&c.\right)chord\,MM',$$

and, dividing by chord MM′,

$$\frac{\textit{surface of cone}\ \mathrm{MM'}}{\textit{chord}\ \mathrm{MM'}}=\pi\left(2y+\frac{dy}{dx}h+\frac{d^2y}{dx^2}\frac{h^2}{1.2}+\&c.\right)$$

If now we represent by s the arc MM′ of the curve, and by u the surface generated by that arc; since, on diminishing h, the arc tends to coincide with the chord, the first side of the preceding equation must be replaced in the case of the limit, by $\frac{du}{ds}$; and the second side being at the same time reduced to $2\pi y$, we shall obtain

$$\frac{du}{ds}=2\pi y,$$

whence $du=2\pi y ds$; and putting for ds its value found, art. 159, we shall have, lastly,

$$du=2\pi y\sqrt{dx^2+dy^2}\ \ldots\ldots\ (78).$$

365. By the method of infinitesimals we should have considered the element of the surface of revolution as that of a truncated cone generated by the revolution of the elementary
Fig. 59. trapezium MPP′M′ (fig. 59) about PP′; and this truncated cone would have for the expression of its surface

$$circ.\frac{(\mathrm{PM}+\mathrm{P\,M'})}{2}\times \mathrm{MM'}=\pi\,(2y+dy)\,ds=2\pi y ds+\pi dy ds,$$

whence, suppressing $\pi dy ds$ as an infinitesimal of the second order, there would remain, for the element of a surface of revolution,

$$2\pi y ds=2\pi y\sqrt{dx^2 dy^2}.$$

366. As a first application, we will take the surface of a paraboloid of revolution, which is the solid generated by the
Fig. 60. revolution of an arc AM (fig. 60) of a parabola about its axis: the equation of the parabola $y^2=px$ gives

$$dx=\frac{2ydy}{p} \text{ and } dx^2=\frac{4y^2dy^2}{p^2};$$

this value being substituted in the formula $2\pi y\sqrt{dx^2+dy^2}$, it is reduced to

$$2\pi y\sqrt{\left(\frac{4y^2+p^2}{p^2}\right)dy^2}=\frac{2\pi}{p}ydy\sqrt{4y^2+p^2},$$

and ydy being the differential of the quantity under the radical sign, except as to the constant, we must put (art. 271) . . . . $4y^2+p^2=z$, when differentiating we find $ydy=\frac{dz}{8}$, and substituting and integrating, we obtain

$$\int\frac{2\pi}{p}ydy\sqrt{4y^2+p^2}=\int\frac{\pi}{4p}z^{\frac{1}{2}}dz=\frac{\pi}{6p}.z^{\frac{3}{2}}+C$$

$$=\frac{\pi}{6p}(4y^2+p^2)^{\frac{3}{2}}+C.$$

The constant is determined by supposing that the integral is 0 when y is 0, which reduces the preceding equation to

$$0=\frac{\pi}{6}p^2+C, \text{ whence } C=-\frac{\pi}{6}p^2;$$

and supposing that the integral is taken from $y=0$ to $y=b$, the definite integral will be

$$\frac{\pi}{6p}[(4b^2+p^2)^{\frac{3}{2}}-p^3]$$

367. As a second application we will find the value of the surface of a sphere. This curve surface being generated by the revolution of the semi-circumference about its diameter, let $x^2+y^2=a^2$ be the equation of the circle: this being differentiated, we shall find,

$$xdx+ydy=0,$$

whence

$$dy = -\frac{x\,dx}{y}, \quad dy^2 = \frac{x^2dx^2}{y^2},$$

and substituting this value in the formula (78), we shall obtain

$$\int 2\pi y\sqrt{\left(\frac{x^2}{y^2}+1\right)dx^2} = \int 2\pi dx\sqrt{x^2+y^2}$$
$$= \int 2\pi a\,dx = 2\pi ax + \text{C} \ldots (79).$$

Fig. 61. To determine the constant, we will take the integral to commence from the point A (fig. 61); and since the origin of the abscissæ is at the centre, we shall suppose the integral to be 0 when $x = -a$, an hypothesis which will change the equation (79) into

$$0 = -2\pi a^2 + \text{C}, \text{ and therefore } \text{C} = 2\pi a^2,$$

and substituting this value in the equation (79), we shall have

$$\int 2\pi a\,dx = 2\pi\,(ax + a^2).$$

Taking now the definite integral betwixt the limits $x = -a$, and $x = a$, we must change x into a in the preceding formula, and we shall obtain for the surface of the sphere,

$$\int 2\pi a\,dx = 2\pi\,(2a^2) = 4\pi a^2.$$

368. We may also find the surface of a cylinder; for this surface being generated by the revolution of the rectangle

Fig. 62. AD (fig. 62) about the axis AB, let $\text{AB} = a$, $\text{AC} = b$; then the equation of the straight line CD will be $y = b$, and therefore $dy = 0$. Substituting these values in the formula (78), it is reduced to $2\pi b\,dx$, and integrating we have

$$\int 2\pi b\,dx = 2\pi bx + \text{C};$$

whence, taking the definite integral betwixt the limits $x = 0$ and $x = a$, we find for the surface of the cylinder

$$2\pi ba = 2\pi b \times a = \textit{circumference of base} \times \textit{height}.$$

In regard to the surface of the cone, this solid being generated by the rotation of the right angled triangle ABC (fig. 55) Fig. 55. about the axis AB, let $AB=a$, $CB=b$; then the equation of AC will be $y=\frac{b}{a}x$, and this being differentiated, gives

$$dy=\frac{b}{a}dx,\ dy^2=\frac{b^2}{a^2}dx^2.$$

These values of y and dy^2 being substituted in the formula (78), we have

$$\int 2\pi y\sqrt{dx^2+dy^2}=\int 2\pi\frac{bx}{a^2}dx\sqrt{a^2+b^2}=\pi\frac{bx^2}{a^2}\sqrt{a^2+b^2}+\mathrm{C};$$

and taking the definite integral betwixt the limits $x=0$ and $x=a$, we obtain

$$\textit{area of cone}\ =\pi b\sqrt{a^2+b^2}=2\pi b\times\frac{\mathrm{AC}}{2}$$

$$=\textit{circumference}\ \mathrm{BC}\times\frac{\mathrm{AC}}{2}.$$

On the cubature of solids of revolution.

369. Let v be the volume of the solid generated by the revolution of the curvilinear area ABMP about the axis AX (fig. 51). Fig. 51.

If the abscissa $AP=x$ become $AP'=x+h$, this volume will be augmented by the part generated by the revolution of the mixtilinear trapezium PMM′P′ about the same axis; and since the volume generated by ABMP is a function of x, for it increases or decreases at the same time with x, the volume generated by ABM′P′ will be a function of $x+h$, and will have for its expression

$$v+\frac{dv}{dx}h+\frac{d^2v}{dx^2}\frac{h^2}{1.2}+\&c.,$$

whence, consequently, subtracting v, the volume generated by ABMP, we shall have

$$\frac{dv}{dx}h+\frac{d^2v}{dx^2}\frac{h^2}{1.2}+\&c.$$

for the volume generated by PMM P′.

But this volume being comprised between the cylinders generated by the two rectangles MP′ and M′P, must differ less from either of those cylinders than the cylinders differ from each other; and if, therefore, it can be proved that, in the case of the limit, the ratio of those cylinders is unity, still more must this be true for the ratio of the volume described by PMM P′ to one of the cylinders. This being premised, we have, evidently,

$$\textit{cylinder described by } \mathrm{PM'}=\pi[f(x+h)]^2h,$$
$$\textit{cylinder described by } \mathrm{P'M}=\pi(fx)^2h;$$

the ratio of those cylinders is, therefore, expressed by

$$\frac{[f(x+h)]^2}{(fx)^2};$$

and since, on making $h=0$, this ratio is obviously reduced to unity, the same will be the case with the ratio of the volume generated by PMM P′ to that of the cylinder described by MP′. But this last ratio being represented by

$$\frac{\frac{dv}{dx}h+\frac{d^2v}{dx^2}\frac{h^2}{2}+\&c.}{\pi(fx)^2h}=\frac{\frac{dv}{dx}+\frac{d^2v}{dx^2}\frac{h}{2}+\&c.}{\pi(fx)^2}$$

we have, in the case of the limit,

$$\frac{\frac{dv}{dx}}{\pi(fx)^2}=1;$$

whence we deduce

$$\frac{dv}{dx} = \pi(fx)^2 = \pi y^2,$$

and lastly

$$dv = \pi y^2 dx \;.\;.\;.\;.\; (80)$$

370. We might arrive at the same result by the consideration of infinitesimals; for the volume MON (fig. 64) may be Fig. 64. conceived to be divided into infinitely thin slices, by planes perpendicular to the axis of revolution; and one of these slices, which will be the element of the solid, may be considered as a cylinder, the base of which is the circle described by y, and its height the thickness ab of the slice represented by dx; this element will consequently be expressed by $\pi y^2 dx$.

371. Applying this formula to the determination of the volume of the prolate spheroid, which is the solid generated by the revolution of an ellipse about its major axis, since the equation of the ellipse referred to the centre is . . . . $y^2 = \frac{b^2}{a^2}(a^2 - x^2)$, we must substitute this value of y^2 in the formula $\pi y^2 dx$, when we shall have

$$\pi y^2 dx = \pi \frac{b^2}{a^2}(a^2 - x^2) dx\,;$$

and integrating, we shall find

$$\int \pi y^2 dx = \pi \frac{b^2}{a^2}\left(a^2 x - \frac{x^3}{3}\right) + C \;.\;.\;.\; (81).$$

Supposing that the integral is 0 at the point A (fig. 56), Fig. 56. where $x = -a$, we shall have

$$C = \pi \frac{b^2}{a^2} \times \frac{2}{3} a^3,$$

and substituting this value of C, the equation (81) becomes

$$\int \pi y^2 dx = \pi \frac{b^2}{a^2}\left(a^2 x - \frac{x^3}{3} + \frac{2}{3}a^3\right).$$

Making, then, $x = a$, in order to have the definite integral comprised between the limits $x = -a$, and $x = +a$, we shall obtain

$$\int \pi y^2 dx = \pi \frac{b^2}{a^2} . \frac{4}{3} a^3 = \frac{4}{3}\pi b^2 a,$$

which is the volume of the prolate spheroid.

If $b = a$, this volume will become that of the sphere, and will have for its expression

$$\frac{4}{3}\pi a^3 = \frac{2}{3}\pi a^2 \times 2a = \frac{2}{3} \text{ of the \textit{circumscribed cylinder}.}$$

We may also determine the volume of the paraboloid of revolution; for which purpose, taking the general parabola as the generating curve, its equation will give

$$y = a x^{\frac{n}{m}},$$

and substituting this value in the formula (80), we shall obtain

$$v = \int \pi a^2 x^{\frac{2n}{m}} dx = \frac{m \pi a^2 x^{\frac{2n+m}{m}}}{2n+m} + C.$$

To determine the constant, we shall suppose that the volume is 0 at the origin where $x = 0$, whence we shall have $C = 0$. In the case of the common parabola, $m = 2$, $n = 1$, and therefore

$$v = \pi a^2 \frac{x^2}{2} = \pi a^2 x . \frac{x}{2} = \pi y^2 . \frac{x}{2}.$$

Fig. 65. Now πy^2 being the area of the circle of which PM (fig. 65) is the radius, the expression $\frac{1}{2}\pi y^2 . x$ represents the half of the

cylinder described by APMB about the axis of the abscissæ; and therefore the volume of the common paraboloid is the half of that of the circumscribed cylinder. (*Note seventh*).

On the cubature of bodies bounded by curve surfaces, by means of double integrals.

372. Let EDCB (fig. 85) be a solid contained within the angle of the coordinate axes Ax, Ay, Az, and terminated by a plane DCG, parallel to the plane of yz; if x become $x+h$, the volume of this solid will be increased by a slice DD′CC′, whose thickness is h; and representing by V′ what the volume then becomes, we shall have Fig. 85.

$$V'=V+\frac{dV}{dx}h+\frac{d^2V}{dx^2}\frac{h^2}{1.2}+\frac{d^3V}{dx^3}\frac{h^3}{1.2.3}+\&c.$$

and the slice DD′CC′FG will be expressed by

$$V'-V=\frac{dV}{dx}h+\frac{d^2V}{dx^2}\frac{h^2}{1.2}+\frac{d^3V}{dx^3}\frac{h^3}{1.2.3}+\&c.;$$

which, in the case of the limit, gives us

$$\frac{V'-V}{h}=\frac{dV}{dx}\ \ldots\ldots\ (81).$$

The two methods of limits and infinitesimals have already been fully explained, and we shall not scruple, therefore, to introduce here certain considerations, derived from the latter, which place the subject before us in a clearer light; after which we can again recur to our method of limits. The equation (81) then shows us that $\frac{dV}{dx}$ is the differential coefficient which determines the volume; the differential consequently is $\frac{dV}{dx}dx$; and this differential is no other than the indefinitely thin slice DD′CC′FG, of which dx is the thickness. If in this slice we make y vary, it will become indefinitely thin in respect of y, as it is already in respect of x; and consequently it will be reduced to an elementary prism ID′KF, the height of which is z, and its base the parallelogram FGKL$=dxdy$; we shall have therefore

$$\frac{d^2V}{dxdy}dxdy=z\,dxdy;$$

whence

$$\frac{d^2V}{dxdy}=z;$$

and replacing z by its value derived from the equation of the curve, this value will be, generally, a function of x and y, which we may represent by M, and we shall have

$$\frac{d^2V}{dxdy}=M.$$

373. To determine the volume from this expression, we shall put it under the form

$$\frac{d\frac{dV}{dx}}{dy}dy=Mdy;$$

when the notation of the first side shows us that the expression for the differential of $\frac{dV}{dx}$ has been arrived at by considering y as variable and x as constant. The same hypothesis, consequently, must hold good when we come to the inverse operation of integrating; in which case x, being considered as constant, may be found in the constant which is to be added to the integral. We shall therefore consider this constant as, generally, a function of x, and representing it by X, we shall have, for a first integration,

$$\frac{dV}{dx}=\int Mdy+X \quad . \quad . \quad . \quad (82).$$

To effect the second integration, we must observe that the notation $\frac{dV}{dx}$ intimates that the differential of the surface has been taken, considering x alone as variable; the same hypothesis consequently must be adhered to in the inverse operation of integrating; so that, representing by Y the function of y which replaces the constant, and multiplying first by dx, in order to change the differential coefficient into the differential, we shall find

$$V=\int dx(\int Mdy+X)+Y.$$

374. The order of the integrations is evidently arbitrary; and the preceding operations may therefore be indicated thus:

$$V=\int\int zdxdy \quad . \quad . \quad . \quad (83).$$

375. To give an application of this method, let it be required to find the volume of the sphere; the equation of the sphere being

$$x^2+y^2+z^2=r^2;$$

we must from this equation deduce the value of z, and substituting it in the formula (83), we shall have

$$\iint z\,dx\,dy \text{ or } \int dy\int z\,dx=\int dy\int dx\sqrt{r^2-x^2-y^2}\ \ldots\ (84);$$

y then being considered as constant, and the difference r^2-y^2, which is essentially positive, being represented by A^2, we shall find, integrating in respect of x,

$$\int dx\sqrt{r^2-x^2-y^2}=\int dx\sqrt{A^2-x^2}:$$

but, from art. 282, we have

$$\int dx\sqrt{A^2-x^2}=\frac{x}{2}\sqrt{A^2-x^2}+\frac{1}{2}A^2\sin^{-1}.\frac{x}{A}+Y;$$

whence, replacing the value of A^2, we find

$$\int dx\sqrt{r^2-x^2-y^2}=\frac{x}{2}\sqrt{r^2-x^2-y^2}+\frac{1}{2}(r^2-y^2)\sin^{-1}\frac{x}{\sqrt{r^2-y^2}}+Y\ ..\ (85).$$

To obtain the definite integral, we must observe that the constant value of y being represented by AP (fig. 86), all the points determined by this equation Fig. 86.
must have their projections in the direction of the line PM: for any one of these points having the variable z for its ordinate will have AQ, QN, for its other coordinates, and QN will be equal to the constant AP, whilst AQ, in the direction of x, may be replaced by PN, so that measuring the values of x along the line PM, the values of y will be constant; taking the integral, therefore, from P to M, that is to say, from $x=0$ to $x=\text{PM}=\sqrt{r^2-y^2}$, we must substitute successively for x, on the second side of the equation (85), the values $x=\sqrt{r^2-y^2}$, $x=0$, and subtracting the second result from the first, we shall find, for the definite integral,

$$\frac{1}{2}(r^2-y^2)\sin^{-1}.\,1.$$

Now the arc whose sine is 1 is equal to the fourth part of the circumference, represented by 2π, and the definite integral therefore becomes

$$\frac{1}{2}(r^2-y^2)\frac{\pi}{2};$$

which value of $\int dx\sqrt{r^2-x^2-y^2}$ being substituted in the equation (84), we shall have

$$\iint z\,dx\,dy=\frac{\pi}{4}\int(r^2-y^2)\,dy=\frac{\pi}{4}\left(r^2y-\frac{y^3}{3}\right)+X;$$

and integrating from $y=0$ to $y=r$, we shall find

$$\int\int z dx dy=\frac{\pi}{4}\left(r^3-\frac{r^3}{3}\right)=\frac{2\pi r^3}{12}=\frac{1}{6}\pi r^3.$$

This will be the volume resting on the quadrant of the circle BAC, and will consequently be the eighth part of the sphere. (*Note sixth*).

On the quadrature of curve surfaces, by means of double integrals.

Fig. 85. 376. Let EDCB (fig. 85) be a curve surface$=$S, and suppose that the abscissa x is increased by h; the surface then will become . . . . $S+\frac{dS}{dx}h+\frac{d^2S}{dx^2}\frac{h^2}{1.2}+$ &c.; and in the case of the limit, the ratio of the increment of the function S to that of the variable x will be reduced to $\frac{dS}{dx}$; the differential therefore will be $\frac{dS}{dx}dx$; and this differential will be represented, in the figure, by the indefinitely narrow strip DD′CC′.

If now y be made to vary, and be taken infinitely small, the strip DD′CC′ will be reduced to DD′II′, and will have for its expression $\frac{d^2S}{dxdy}dxdy$.

But the surface DD′II′ being indefinitely small, it may be considered as a plane; and, consequently, when multiplied by the cosine of its inclination γ to the plane of xy, it will be equal to $dxdy$ (*note seventh*); and we shall have, therefore,

$$\text{DD'II'}\cos\gamma=dxdy,$$

or

$$\frac{d^2S}{dxdy}dxdy\cos\gamma=dxdy,$$

whence we deduce

$$\frac{d^2S}{dxdy}=\frac{1}{\cos\gamma}.$$

To determine the value of γ, let $Ax+By+Cz+D=0$ be the equation of the plane tangent; we know then that this plane makes with the plane y, an angle which is given by the equation (*note eighth*).

$$\cos\gamma=\sqrt{1+\left(\frac{dz}{dy}\right)^2+\left(\frac{dz}{dx}\right)^2};$$

considering, therefore, $Ax+By+Cz+D=0$ as the equation of the tangent plane at the point of the curve surface whose projection is *dxdy*, we shall have

$$\frac{d^2S}{dxdy}=\sqrt{1+\left(\frac{dz}{dy}\right)^2+\left(\frac{dz}{dx}\right)^2}\ldots\ (86).$$

To determine the differential coefficients which enter into this expression, we must observe that at the point under consideration the tangent plane coincides with the curve surface, the equation to which we shall represent by $z=f(x,y)$; and, consequently, the values of $\frac{dz}{dy}$ and $\frac{dz}{dx}$, which enter into the expression for cos γ, must be regarded (art. 75) as the same with those deduced immediately from the equation $z=f(x, y)$. Before making these substitutions, the equation (86), multiplied by $dxdy$, must be integrated twice in order, an operation which we shall indicate, as before, by the double sign of integration, and we shall have

$$S=\int\int dxdy\sqrt{1+\left(\frac{dz}{dy}\right)^2+\left(\frac{dz}{dx}\right)^2}.$$

377. To give an application of this formula, we will determine the expression for the surface of the sphere.

Its equation being

$$x^2+y^2+z^2=r^2\ \ldots\ (87),$$

we must differentiate it, and we shall find, after dividing by 2,

$$xdx+ydy+zdz=0,$$

whence we deduce

$$dz=-\frac{x}{z}dx-\frac{y}{z}dy;$$

and consequently

$$\frac{dz}{dx}=-\frac{x}{z},\ \frac{dz}{dy}=-\frac{y}{z}.$$

Substituting these values in the expression

$$\sqrt{1+\left(\frac{dz}{dy}\right)^2+\left(\frac{dz}{dx}\right)^2},$$

we shall change it into

$$\sqrt{1+\frac{x^2}{z^2}+\frac{y^2}{z^2}}=\frac{1}{z}\sqrt{z^2+x^2+y^2}=\frac{r}{z},$$

and consequently

$$\int\int dxdy\sqrt{1+\left(\frac{dz}{dx}\right)^2+\left(\frac{dz}{dy}\right)^2}=\int\int\frac{rdxdy}{z};$$

and putting the value of z, we shall have

$$\int\int dxdy\sqrt{1+\left(\frac{dz}{dx}\right)^2+\left(\frac{dz}{dy}\right)^2}=\int\int\frac{rdxdy}{\sqrt{r^2-x^2-y^2}}.$$

378. To effect these integrations we shall put

$$\frac{\int\int rdxdy}{\sqrt{r^2-x^2-y^2}}=\int rdy\int\frac{dx}{\sqrt{r^2-x^2-y^2}}\quad\ldots\ (88),$$

marking thereby that we are to commence with integrating the expression $\frac{dx}{\sqrt{r^2-x^2-y^2}}$, considering x as the only variable.

Making, therefore, as before, $r^2-y^2=A^2$, and integrating, according to art. 274, we shall have, adding a constant function of y,

$$\int\frac{dx}{\sqrt{A^2-x^2}}=\sin^{-1}\frac{x}{A}+Y;$$

putting, then, for A its value, and taking the definite integral from $x=0$ to $x=\sqrt{r^2-y^2}$, there will result

$$\int\frac{dx}{\sqrt{r^2-x^2-y^2}}=\sin^{-1}.1=\frac{1}{4}\textit{circumference}=\frac{\pi}{2};$$

and this value being substituted in equation (88), will give us

$$\int\int\frac{rdxdy}{\sqrt{r^2-x^2-y}}=\int\frac{\pi}{2}rdy=\frac{1}{2}\pi ry+X,$$

where X represents the constant which must be considered as a function of x; taking, then, the definite integral between the limits $y=0$ and $y=r$, we shall find, lastly,

$$\int\int\frac{rdxdy}{\sqrt{r^2-x^2-y^2}}=\frac{1}{2}\pi r^2.$$

The surface thus determined will be the part comprised within the angle formed by the axis of the rectangular coordinates x, y, z, that is to say, one-eighth of the surface of the sphere.

On the integration of functions of two variables.

379. The two principal methods employed for arriving at the integration of differential equations, which contain two or a greater number of variables, consist, 1st, in the separation of the variables, in order to apply to them the usual processes for a single variable; 2d, in the investigation of factors proper to render a differential complete. We shall proceed now to the discussion of these two methods.

On the separation of the variables, the linear equation of the first order, and the properties of homogeneous functions.

380. It will be seen that every differential, to be integrable by the rules already given, must be of the form $\varphi x dx$; so that we should be at a loss how to effect the integration, should the equation contain terms such as y^2dx, $xydx$, $\frac{dx}{y}$, &c. We are not, however, to conclude that the integration is impracticable; for if, by algebraical operations, the expression can be so transformed that each term shall contain only one variable, the integration may still be effected. The equation $xdy+ydx=0$ comes under this case; for by dividing this equation by xy, it becomes

$$\frac{dy}{y}+\frac{dx}{x}=0;$$

which, being integrated, gives

$$\log y+\log x=\text{C},$$

and representing by A the number of which C is the logarithm, we have

$$\log y+\log x=\log \text{A};$$

whence, consequently,

$$\log xy=\log \text{A},$$

and, passing to numbers, we have

$$xy = \mathrm{A}.$$

381. Let the expression be the general equation

$$\phi x . dy + \mathrm{F}y . dx = 0\,;$$

to separate the variables, we must divide each term by $\varphi x . \mathrm{F}y$, and we shall find

$$\frac{dy}{\mathrm{F}y} + \frac{dx}{\varphi x} = 0,$$

an equation in which the variables are separated.

382. To give an example, let it be proposed to integrate

$$(1+x^2)\, dy = dx \sqrt{y}:$$

dividing by $(1+x^2)\sqrt{y}$, we shall have

$$\frac{dy}{\sqrt{y}} = \frac{dx}{1+x^2}\,;$$

and integrating this equation, we shall obtain

$$2\sqrt{y} = \tan^{-1} x + \mathrm{C}.$$

383. The variables may be separated also by division in the formula

$$\varphi x . \mathrm{F}y . dx + \varphi' x . \mathrm{F}' y . dy = 0.$$

For this purpose we have only to divide by $\mathrm{F}y . \phi' x$, when we shall have

$$\frac{\varphi x}{\varphi' x} dx + \frac{\mathrm{F}' y}{\mathrm{F}y} dy = 0.$$

This process is applicable to the equation

$$x^2 y dx + (3y+1)\, dy \sqrt{x^3} = 0,$$

for, if we divide by $y \sqrt{x^3}$, we obtain

$$\frac{x^2}{\sqrt{x^3}}dx+\frac{3y+1}{y}dy=0.$$

384. The integration may also be effected, if the proposed equation involving two variables can be so reduced that each side shall contain only differentials of which the integrals are known; as, for instance, the functions

$$\frac{ydx-xdy}{y^2},\ xdy+ydx,\ \&\text{c.},$$

the integrals of which are respectively $\frac{x}{y}$ and xy.

385. There is an important equation in which the separation of the variables is effected in a manner particularly ingenious.

This equation is

$$dy+\mathrm{P}ydx=\mathrm{Q}dx\ \ldots\ldots\ (89),$$

where P and Q are functions of x, and it is integrated thus: y is assumed equal to the product of two indeterminate quantities, X and z, which gives

$$y=\mathrm{X}z,\ dy=\mathrm{X}dz+zd\mathrm{X}\ ;$$

these values being substituted in the equation (89), it is transformed into

$$zd\mathrm{X}+\mathrm{X}\,(dz+\mathrm{P}zdx)=\mathrm{Q}dx\ ;$$

and the function X being arbitrary, it is determined by equating to each other the terms without the brackets; which resolves the preceding equation into the two

$$\mathrm{X}\,(dz+\mathrm{P}zdx)\ =0,\ zd\mathrm{X}=\mathrm{Q}dx.$$

The first of these gives

$$\frac{dz}{z}=-\mathrm{P}dx,\ \text{whence}\ \log z=-\!\int\!\mathrm{P}dx,$$

or, observing that $\log e$ is equivalent to unity,

$$\log z = -\int Pdx \log e = \log e^{-\int Pdx},$$

and, therefore,

$$z = e^{-\int Pdx}:$$

from the second we deduce

$$dX = \frac{Qdx}{z,} = Q.e^{\int Pdx}dx;$$

whence

$$X = \int Qe^{\int Pdx}dx + C;$$

and putting these values of z and X in the equation

$$y = zX,$$

we obtain

$$y = e^{-\int Pdx}(\int Qe^{\int Pdx}dx + C) \quad \ldots\ldots (90).$$

This equation bears the name of the *linear equation of the first order;* the reason will be seen, art. 445.

386. The separation of the variables may always be effected in differential equations of the first order and betwixt two variables, when the equations are homogeneous. An homogeneous equation is one in which all the terms, considered in respect to the variables, are of the same dimensions; thus the equation

$$ax^2y^3 + bxy^4 + cy^3x^2 = 0$$

is homogeneous, since the sum of the indices of x and y in each term is equal to 5, and the products x^2y^3, xy^4, y^3x^2, are each of five dimensions.

The equation

$$ax^6y^2 - bx^5y^3 + cy^8 = 0$$

is also homogeneous, since the sum of the indices of the variables in each term being 8. The variable x does not appear in the last term of the equation, but it may be considered as having an index 0.

387. Let, generally, z be a function of x and y, composed of homogeneous terms, such as Ax^py^q, $Bx^{p'}y^{q'}$, $Cx^{p''}y^{q''}$, &c. If we represent by n the sum of the indices of x and y, in one of these terms, we shall have, by virtue of their homogeneous nature,

$$p+q=n,\ p'+q'=n,\ p''+q''=n, \text{ \&c.}$$

If now we divide all the terms by x^n, this equality will still subsist, and the term Ax^py^q will in this case become

$$\frac{Ax^py^q}{x^n}=\frac{Ay^q}{x^{n-p}}=\frac{Ay^q}{x^q}=A\left(\frac{y}{x}\right)^q;$$

and since what has been said of this term will apply to all the rest, we shall have

$$\frac{z}{x^n}=\Gamma\left(\frac{y}{x}\right):$$

or, making $\frac{y}{x}=q$,

$$x^nFq=z,$$

and the function Fq being represented by Q, this equation may be written thus:

$$Qx^n=z.$$

388. We shall now take into consideration the differential equation

$$Mdx+Ndy=0,$$

in which the coefficients M and N are homogeneous functions of two variables x and y, and of a dimension n.

This equation being divided by x^n, it will, as we have seen, result under the form

$$\varphi\left(\frac{y}{x}\right)dx+\mathrm{F}\left(\frac{y}{x}\right)dy=0\,;$$

and making $\frac{y}{x}=z$, this equation will become

$$dx\varphi z+dy\mathrm{F}z=0,$$

or

$$\varphi z+\mathrm{F}z\frac{dy}{dx}=0\;.\;.\;.\;(91).$$

In order to eliminate y by means of the equation $\frac{y}{x}=z$, or . . . $y=zx$, we must differentiate this latter equation, when we shall obtain

$$\frac{dy}{dx}=z+\frac{xdz}{dx},$$

which reduces the equation (91) to

$$\varphi z+\mathrm{F}z\left(z+\frac{xdz}{dx}\right)=0\,;$$

from this we deduce

$$\frac{xdz}{dx}=-\frac{\varphi z}{\mathrm{F}z}-z=-\frac{(\varphi z+z\mathrm{F}z)}{\mathrm{F}z},$$

and, separating the variables,

$$\frac{dx}{x}=-\frac{dz\mathrm{F}z}{\varphi z+z\mathrm{F}z},$$

and consequently

$$\log x=-\int\frac{dz\mathrm{F}z}{\varphi z+z\mathrm{F}z}+\mathrm{C}.$$

The integration being completed, it will remain only to substitute in the result the value of z.

389. We will take, for example, the equation . . . . . . $x^2dy=y^2dx+xydx$; when, making $y=zx$, we shall find

$$dy=zdx+xdz,$$

and substituting these values, the equation will become

$$x^2zdx+x^3dz=z^2x^2dx+zx^2dx;$$

reducing and dividing by the common factor x^2, we shall obtain

$$xdz=z^2dx,$$

which equation being divided by z^2x, gives

$$\frac{dx}{x}=\frac{dz}{z^2},$$

and integrating, we shall have

$$\log x=-\frac{1}{z}+C=-\frac{1}{\frac{y}{x}}+C=-\frac{x}{y}+C.$$

390. Our second example shall be the equation

$$\frac{x^2+yx}{x-y}dy=ydx,$$

the denominator in which being made to disappear, we see that all the terms will be of two dimensions, and we must, therefore, assume $y=zx$; when, being reduced, the equation will give us

$$\frac{dy}{dx}=z\frac{(1-z)}{(1+z)},$$

and putting for $\frac{dy}{dx}$ its value derived from the equation $y=zx$, we shall have

$$z+\frac{xdz}{dx}=z\frac{(1-z)}{(1+z)};$$

z being then transposed to the second side, and that side reduced to a common denominator, we shall find

$$\frac{dx}{x}=-\frac{(1+z)}{2z^2}dz,$$

and, lastly,

$$\log x=-\int\frac{dz}{2z^2}-\int\frac{dz}{2z}=\frac{1}{2z}-\tfrac{1}{2}\log z+\mathrm{C}$$

$$=\frac{x}{2y}-\tfrac{1}{2}\log\frac{y}{x}+\mathrm{C}.$$

391. When the proposed equation, besides the terms $\mathrm{A}x_py_q$, $\mathrm{B}x^{p'}y^{q'}$, &c. contains polynomials such as

$$(\mathrm{M}x^ry^s+\mathrm{N}x^{r'}y^{s'}+\&c.)_kdx,\ (\mathrm{P}x^ty^u+\mathrm{Q}x^{t'}y_{u'}+\&c.)_ldy,$$

the variables will be still separable if we have

$$p+q=p'+q'=(r+s)k=(r'+s')k=(t+u)l=(t'+u')l\ \ldots\ (92).$$

To prove this, let

$$(r+s)k=n,\ (r'+s')k=n\ \ldots\ (93),$$

and divide all the terms of the polynomial $(\mathrm{M}x^ry^s+\mathrm{N}x^{r'}y^{s'}+\&c.)^k$ by x^n, when it will become

$$\left(\frac{\mathrm{M}x^ry_s+\mathrm{N}x_{r'}y_{s'}+\&c.}{x^{\frac{n}{k}}}\right)^k=\left(\frac{\mathrm{M}y^s}{x^{\frac{n}{k}-r}}+\frac{\mathrm{N}y_{s'}}{x^{\frac{n}{k}-r'}}+\&c.\right)^k;$$

but the equations (93) give us

$$\frac{n}{k}-r=s,\ \frac{n}{k}-r'=s',$$

and, therefore, substituting these values in the preceding expression, we shall find

$$\left(\frac{My^s}{x^s}+N\frac{y^{s'}}{x^{s'}}.+\&c.\right)^k=\left[M\left(\frac{y}{x}\right)^s+N\left(\frac{y}{x}\right)^{s'}+\&c.\right]^k,$$

which proves that when the equations (92) are satisfied, the polynomials raised to any powers, reduce themselves, as the other terms, to functions of $\frac{y}{x}$; and, consequently, making $\frac{y}{x}=z$, or $y=xz$, the equation may be reduced to a function of z. To give an example, let

$$xdy-ydx=dx\sqrt{x^2-y^2}\quad .\ .\ .\ .\ (94).$$

This equation being written thus,

$$x^1y^0dy-y^1x^0dx=dx(x^2y^0-y^2x^0)^{\frac{1}{2}},$$

we see that the equations (92) are satisfied; we shall, therefore, assume $y=xz$, and consequently

$$\frac{dy}{dx}=z+x\frac{dz}{dx};$$

substituting these values in the equation (94), and reducing and dividing by the common factor, we shall obtain

$$x\frac{dz}{dx}=\sqrt{1-z^2},$$

whence

$$\frac{dx}{x}=\frac{dz}{\sqrt{1-z^2}},$$

and integrating, we shall find (art. 273),

$$\log x=\sin^{-1}z+C,$$

or, replacing the value of z,

$$\log x=\sin^{-1}.\frac{y}{x}+C.$$

392. Generally, when we have an homogeneous function of the variables x, y, z, &c., we may always separate one of the variables, for instance, x, by making $y=tx$, $z=ux$, &c.*

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* Let $Mdx+Ndy+Pdz=0$ be an homogeneous function, in which M, N, P, are functions of the three variables x, y, z; these functions

393. We sometimes employ indeterminate indices for the purpose of rendering an equation homogeneous; let the equation, for example, be

$$ay^mx^ndx+bx^pdx+cx^qdy=0;$$

we shall assume $y=z^k$; and since the index k is not a variable, but an unknown constant, we may differentiate by the art. 18, when we shall have

$$dy=kz^{k-1}dz, \text{ and } y^m=z^{km};$$

and substituting, we shall obtain

$$az^{km}x^ndx+bx^pdx+ckx^qz^{k-1}dz=0,$$

an equation which will be homogeneous if we have

$$km+n=p,\ q+k-1=p.$$

M, N, P, will contain terms such as $Ax^py^qz^r$, $Bx^{p'}y^{q'}z^{r'}$, $Cx^{p''}y^{q''}z^{r''}$, and we shall have $p+q+r=p'+q'+r'=p''+q''+r''=n$. If in one of these terms, for instance, in $Ax^py^qz^r$, we substitute the values $y=tx$, $z=ux$, this term will become

$$Ax^pt^qx^qu^rx^r=x^{p+q+r}\times At^qu^r=x^nAt^qu^r;$$

and the same being the case for the other terms, if we substitute the values of y and z, the equation $Mdx+Ndy+Pdz=0$ will have x^n for a common factor, which, being suppressed, the equation will take the form

$$\varphi(t,u)x+F(t,u)d.tx+f(t,u)d.ux=0:$$

and, performing the differentiations, we shall have

$$\varphi(t,u)dx+F(t,u)(tdx+xdt)+f(t,u)(udx+xdu=0;$$

whence we shall deduce

$$[\varphi(t,u)+tF(t,u)+uf(t,u)]dx=-x[F(t,u)dt+f(t,u)du];$$

and consequently

$$\frac{dx}{x}=-\frac{F(t,u)dt+f(t,u)du}{\varphi(t,u)+tF(t,u)+uf(t,u)}.$$

Now eliminating the indeterminate k, we shall find

$$\frac{p-n}{m}=p+1-q,$$

and this, therefore, is the equation of condition that must be satisfied in order that the proposed expression may become homogeneous by the substitution of $y=z^k=z^{p+1-q}$.

394. There is an important theorem, in respect to homogeneous functions, which we shall proceed to demonstrate in the manner following:

Let $Mdx+Ndy$ be the differential of an homogeneous function z betwixt two variables x and y, in which n is the sum of the indices of the variables, in one of the terms composing the function; we shall have then the equation

$$Mdx+Ndy=dz \quad . \; . \; . \quad (95);$$

and making $\frac{y}{x}=q$, we shall find, art 387,

$$Qx^n=z;$$

replacing, in the equation (95), y by its value qx, and representing by M' and N' what M and N then become, the equation (95) is transformed into

$$M'dx+N'd.qx=d.Qx^n \quad . \; . \; . \quad (96),$$

or, putting for $d.qx$ its value, $qdx+xdq$,

$$(M'+N'q)dx+N'xdq=d(Qx^n).$$

But $(M'+N'q)dx$ is the differential of Qx^n, taken in respect of x, so that we have

$$M'+N'q=nQx^{n-1};$$

and putting in this equation y in place of qx, and consequently M, N, in place of M', N', it becomes

$$M+N\frac{y}{x}=nQx^{n-1},$$

or,

$$Mx + Ny = nQx^n = nz.$$

395. This theorem will apply to homogeneous functions of any number of variables; for if we had, for instance, the equation

$$Mdx + Ndy + Pdt = dz,$$

in which the dimension of each term is n, we should only have to make $\frac{y}{x} = q$, $\frac{t}{x} = r$, to prove, by reasoning analogous to what has been just employed, that we must have $z = x^n F(q, r)$, and of course

$$Mx + Ny + Pt = nz.$$

The conditions of integrability of functions of two variables.—Integration of functions which fulfil those conditions.—Investigation of factors proper to render equations integrable which are not immediately so.

396. When we have a differential $Mdx + Ndy = 0$, we cannot conclude that there is always some equation, which, being differentiated, will give the proposed one; for if, for instance, we had differentiated the equation $f(x, y) = 0$, and derived from it $mdx + ndy = 0$, we might multiply this equation by a function of x, and so obtain an equation $Mdx + Ndy = 0$, in which the coefficients M and N are different from m and n; and consequently the equation

$$Mdx + Ndy = 0$$

would no longer be the result of simply differentiating the function

$$f(x, y) = 0.$$

The same would be the case, if we should combine arbitrarily $mdx + ndy = 0$ with the primitive equation $f(x, y) = 0$: for example, by eliminating one or more terms betwixt $mdx + ndy = 0$ and $f(x, y) = 0$, we might obtain an equation

$$M'dx + N'dy = 0,$$

in which the differentials M′ and N′ differ from m and n.

397. An equation which, like $mdx + ndy = 0$, has been obtained by the process of differentiation alone, is named a complete differential; as is also every differential function which has been found by means of differentiation only, though it be not equal to zero.

When a differential equation $Mdx + Ndy = 0$ is not a complete differential, we must not think of integrating it, until by some preparatory operation it has been rendered complete.

398. Euler first resolved this important problem:

1°. *A differential equation being given, how can it be discovered when it is a complete differential?*

2°. *What are the means of integrating this equation?*

Before giving the solution of this problem, we shall call to mind that, according to the notation agreed on, art. 52, the expression $\frac{dz}{dx}$ indicates that the function z of x and y has been differentiated in respect of x, and divided by dx * ; if, then, this function $\frac{dz}{dx}$ is differentiated in respect of another variable y and divided by dy, the result of this operation is written

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* Let $dz = Adx + Bdy + Cdt$ be the complete differential of z; the ratio $\frac{dz}{dx}$ is no other than the differential coefficient A. If, therefore, we were asked the ratio of $Adx + Bdy + Cdt$ to dx, it would not be right to represent it by $\frac{dz}{dx}$; in this case, the ratio of the complete differential to dx might be written in one of the following ways:

$$\frac{1}{dx}dz, \text{ or } \frac{d(z)}{dx}.$$

thus: $\frac{d^2z}{dxdy}$. If, on the contrary, we had taken, first, the differential coefficient of z in respect of y, and then in respect of x, the result of these operations would have been written thus: $\frac{d^2z}{dydx}$.

When z is a function of three variables x, y, u, an expression such as $\frac{d^3z}{dxdydu}$ indicates that we have taken first the differential coefficient of z in respect of x, then the differential coefficient of $\frac{dz}{dx}$ in respect of y, and, lastly, the differential coefficient of $\frac{d^2z}{dxdy}$ in respect of u. Similarly the expression $\frac{d^5z}{dx^2dy^3}$ indicates that we have effected five successive differentiations of z, the two first in respect of x, and the three others in respect of y.

399. This being premised, Euler's theorem rests on the following proposition, which has been demonstrated, art. 172.

If we have a function z *of two variables* x *and* y, *and we take first the differential coefficient of* z *in respect of* x, *and take then the differential coefficient of* $\frac{dz}{dx}$ *in respect of* y, *we shall have the same result as if we had taken first the differential coefficient of* z *in respect of* y, *and then the differential coefficient of* $\frac{dz}{dy}$ *in respect of* x, *a proposition which we express by saying that*

$$\frac{d^2z}{dxdy}=\frac{d^2z}{dydx}.$$

400. If we have, for example,

$$z=x^2+xy,$$

we find

$$\frac{dz}{dx}=2x+y,\ \frac{dz}{dy}=x,$$

and, consequently,

$$\frac{d^2z}{dxdy}=1=\frac{d^2z}{dydx}.$$

401. This being premised, let z be the function of which the complete differential is $Mdx+Ndy=0$, we have then

$$M=\frac{dz}{dx},\ N=\frac{dz}{dy}.$$

The first of these equations, being differentiated in respect to y, will give

$$\frac{dM}{dy}=\frac{d^2z}{dxdy};$$

the second, being differentiated, in respect of x, will give

$$\frac{dN}{dx}=\frac{d^2z}{dydx};$$

and the second sides of these equations being identical, it follows that

$$\frac{dM}{dy}=\frac{dN}{dx}\ \ldots\ldots(97);$$

whenever therefore this equation of condition is fulfilled, the differential proposed will be complete.

402. We recognize, for instance, that the expression .... $(2x-y)\,dx-xdy$ is a complete differential, because

$$\frac{dM}{dy}=-1=\frac{dN}{dx}.$$

The expression

$$(y^2+3x^2)\,dx+\ (3y^2+2xy)\,dy$$

is also a complete differential, for

$$\frac{dM}{dy}=2y=\frac{dN}{dx}.$$

403. The equation $ydx-xdy=0$ is not a complete differential, since $\frac{dM}{dy}=1$, and $\frac{dN}{dx}=-1$. This equation, in fact, is derived from the one,

$$\frac{ydx-xdy}{y^2}=0,$$

found immediately by differentiation, and in which the common divisor y^2 has been suppressed; restoring it, we shall have

$$M=\frac{1}{y},\ N=-\frac{x}{y^2},$$

and the condition $\frac{dM}{dy}=\frac{dN}{dx}$ will be fulfilled.

404. Let it be proposed now to integrate a differential between two variables, when it has been found that the differential is complete. For this purpose we must observe, first, that when a function z of x and y has given, by differentiation, $Mdx+Ndy$, the term Mdx has been obtained by considering y as constant. Consequently, when we integrate the part Mdx, the constant added may involve y, and representing it therefore by Y (admitting at the same time that, if requisite, Y may be considered as an ordinary constant), we shall write

$$u=\int Mdx+Y=0 \quad . \ . \ . \ . \quad (98).$$

This equation being the one which, by the differentiation, ought to give us $Mdx+Ndy=0$, it follows that N is no other than the differential coefficient of $\int Mdx+Y$, in respect of y. Differentiating on this supposition, we shall have

$$N=\frac{d\int Mdx}{dy}+\frac{dY}{dy}:$$

from this equation we deduce

$$\frac{dY}{dy}=N-\frac{d\int Mdx}{dy},$$

and integrating,

$$Y=\int\left(N-\frac{d\int Mdx}{dy}\right)dy;$$

and this value of Y being substituted in the equation (98), we shall obtain

$$u=\int Mdx+\int\left(N-\frac{d\int Mdx}{dy}\right)dy \;\ldots\ldots\; (99).$$

It may be observed that $N-\frac{d\int Mdx}{dy}$ does not contain x, since this expression, multiplied by dy, must give for its integral a function Y of the variable y alone.

405. To demonstrate that the expression $N-\frac{d\int Mdx}{dy}$ is not a function of x, we shall take its differential coefficient in respect of x, when we shall have

$$\frac{dN}{dx}-\frac{d(d\int Mdx)}{dydx} \;\ldots\; (100),$$

and changing the order of the differentiations, the second part of this expression will become

$$\frac{d(d\int Mdx)}{dxdy}=\frac{d\left(\frac{d\int Mdx}{dx}\right)}{dy}.$$

But the integral $\int Mdx$ having been taken in respect of x, the differential of $\int Mdx$, relatively to the same variable x, will be Mdx, and consequently $\frac{d\int Mdx}{dx}=M$, which reduces the expression $\frac{d\left(\frac{d\int Mdx}{dx}\right)}{dy}$ to $\frac{dM}{dy}$; and this value being substituted in the expression (100), we shall have $\frac{dN}{dx}-\frac{dM}{dy}$, a quantity which, according to the condition of integrability, is 0; it follows, therefore, that the differential of $N - \frac{d\int M dx}{dy}$ in respect of x is 0, and this proves that the expression does not contain x.

406. By means of the formula (98), we may integrate every function of two variables which satisfies the condition of integrability.

Let us take, for example,

$$(6xy-y^2)dx+(3x^2-2xy)dy \quad . \ . \ . \quad (101).$$

Comparing this expression with the formula $Mdx+Ndy$, we have

$$6xy-y^2=M, \quad 3x^2-2xy=N\ ;$$

and the condition of integrability is consequently fulfilled, since we find

$$\frac{dM}{dy}=6x-2y=\frac{dN}{dx}\ ;$$

integrating therefore the expression $(6xy-y^2)dx$ on the supposition of y being constant, we shall have

$$\int Mdx=\int(6xy-y^2)dx=3x^2y-y^2x\ ;$$

and substituting this value and that of N in the equation (99), we shall obtain

$$u=3x^2y-y^2x+\int\left[3x^2-2xy-\frac{d(3x^2y-y^2x)}{dy}\right]dy.$$

The differentiation being performed, the part affected by the sign of integration in this expression is reduced to

$$\int(3x^2-2xy-3x^2+2yx)\,dy\ ;$$

in which the terms within the brackets destroy each other; and it follows, consequently, that the expression represented by

$$\int\left[3x^2-2xy-\frac{d(3x^2y-y^2x)}{dy}\right]dy$$

is constant, since every quantity, whose differential is 0, is constant; the integral sought, therefore, is

$$3x^2y-y^2x+\textit{constant}.$$

407. Without employing the formula found in the preceding article, we might arrive directly at the same result in the manner following:

Integrating the expression (101) on the supposition that y is constant, we shall have

$$\int Mdx=\int(6xy-y^2)dx+Y,$$

or,

$$u=3x^2y-y^2x+Y \quad . \; . \; . \quad (103);$$

and, differentiating this equation in respect of y, we shall obtain

$$\frac{du}{dy}=3x^2-2xy+\frac{dY}{dy} \quad . \; . \; . \quad (104).$$

But $\frac{du}{dy}$ being no other than the coefficient of dy in the expression (101), we have also

$$\frac{du}{dy}=3x^2-2xy,$$

and comparing these two values, we shall find

$$\frac{dY}{dy}=0;$$

and consequently

$$Y=\textit{constant};$$

which value being substituted in the equation (103), we shall have

$$u=3x^2y-y^2x+\textit{constant}.$$

408. Let the function be

$$(2y^2x+3y^3)dx+(2x^2y+9xy^2+8y^3)dy;$$

if we compare this expression with $Mdx+Ndy$, we shall find

$$M=2y^2x+3y^3,\quad N=2x^2y+9xy^2+8y^3\ ;$$

and since we have

$$\frac{dM}{dy}=4yx+9y^2=\frac{dN}{dx},$$

the proposed function is a complete differential.

Integrating therefore in respect of x, we shall have

$$\int Mdx=y^2x^2+3y^3x+Y,$$

or,

$$u=y^2x^2+3y^3x+Y\ ;$$

and differentiating this expression in respect of y, we shall obtain

$$\frac{du}{dy}=\frac{d.(y^2x^2+3y^3x)}{dy}+\frac{dY}{dy}.$$

On the other hand, $\frac{du}{dy}$ being the coefficient of dy in the proposed equation, we have also

$$\frac{du}{dy}=2x^2y+9xy^2+8y^3\ ;$$

and, equating these two values of $\frac{du}{dy}$, we deduce

$$\frac{d\,(y^2x^2+3y^3x)}{dy}+\frac{dY}{dy}=2x^2y+9xy^2+8y^3\ ;$$

whence, performing the differentiation in respect to y, we have

$$2x^2y+9y^2x+\frac{dY}{dy}=2x^2y+9y^2x+8y^3\ ;$$

an equation which reduces itself to

$$\frac{dY}{dy}=8y^3,$$

and therefore

$$Y = \int 8y^3 dy = 2y^4 + C;$$

and consequently the integral sought is

$$u = y^2x^2 + 3y^3x + 2y^4 + C.$$

409. We saw, art. 403, that the equation $ydx - xdy = 0$ was not a complete differential, because it had lost the common factor y^2; it appears, therefore, that there may be equations which, like this, are not immediately integrable, but may become so if the common factor that has been lost can be restored.

410. Let, generally, $Pdx + Qdy = 0$ be the equation which is a complete differential, and z the common factor, which, for greater generality, we shall suppose a function of x and y: we shall have then

$$P = Mz, \quad Q = Nz;$$

and if we substitute these values in the preceding equation, the common factor z will disappear, and we shall have

$$Mdx + Ndy = 0 \quad . \; . \; . \quad (105).$$

Now the equation $Pdx + Qdy = 0$ being, by hypothesis, a complete differential, we must have

$$\frac{dP}{dy} = \frac{dQ}{dx};$$

putting for P and Q their values, this equation will become

$$\frac{dMz}{dy} = \frac{dNz}{dx};$$

and, developing, we shall find

$$\frac{Mdz}{dy} + \frac{zdM}{dy} = \frac{Ndz}{dx} + \frac{zdN}{dx} \quad . \; . \; . \quad (106).$$

411. When the common factor z is constant, $\frac{dz}{dy}$ and $\frac{dz}{dx}$

being 0, the equation (106) becomes

$$\frac{d\mathrm{M}}{dy}=\frac{d\mathrm{N}}{dx};$$

and, consequently, the condition necessary that the equation (105) may be a complete differential is fulfilled. But when z is a function of x and y, the determination of z depends on equation (106); and this equation is more difficult to integrate than the proposed one, which contains only the single differential coefficient $\frac{dy}{dx}$, whilst the equation (106) contains three variables, x, y, z, and the two differential coefficients $\frac{dz}{dx}$ and $\frac{dz}{dy}$.

412. If the equation be homogeneous, it is very easy to determine the factor; for let $\mathrm{M}dx+\mathrm{N}dy=0$ be an homogeneous equation which becomes integrable when multiplied by an homogeneous function z of x and y; representing the integral of the equation $\mathrm{M}dx+\mathrm{N}dy=0$ by u, we have

$$z\mathrm{M}dx+z\mathrm{N}dy=du \quad . \;.\; . \quad (107),$$

and this equation being homogeneous, we deduce, art. 394,

$$z\mathrm{M}x+z\mathrm{N}y=nu \quad . \;.\; . \quad (108).$$

If now the dimension of M be represented by m, and that of z by k, the dimension of one of the terms $z\mathrm{M}x$ or $z\mathrm{N}y$ will be $m+k+1$; this value, therefore, being put in place of n, in the preceding equation, we shall have

$$z\mathrm{M}x+z\mathrm{N}y=(m+k+1)u,$$

and dividing the equation (107) by this, we shall find

$$\frac{\mathrm{M}dx+\mathrm{N}dy}{\mathrm{M}x+\mathrm{N}y}=\frac{du}{u}\times\frac{1}{m+k+1}.$$

The second side of this equation is a complete differential, and the first must, therefore, be so also; whence it follows that $\frac{1}{Mx+Ny}$ is the factor proper to render the homogeneous equation $Mdx+Ndy=0$ integrable.

413. If the common factor z, which ought to render the proposed equation homogeneous, be a function of x alone, we have $\frac{dz}{dy}=0$, which reduces the equation (106) to

$$\frac{zdM}{dy}=\frac{Ndz}{dx}+z\frac{dN}{dx},$$

whence we deduce

$$\frac{Ndz}{dx}=z\left(\frac{dM}{dy}-\frac{dN}{dx}\right),$$

and consequently

$$\frac{dz}{z}=\left(\frac{\frac{dM}{dy}-\frac{dN}{dx}}{N}\right)dx \ldots (109);$$

integrating, therefore, we have

$$\log z=\int\left(\frac{\frac{dM}{dy}-\frac{dN}{dx}}{N}\right)dx$$
$$=\int\frac{1}{N}\left(\frac{dM}{dy}-\frac{dN}{dx}\right)dx;$$

and multiplying by $\log e$, making the coefficient of $\log e$ the index, and passing to numbers, we find

$$z=e^{\int\frac{1}{N}\left(\frac{dM}{dy}-\frac{dN}{dx}\right)dx} \ldots (110).$$

We have only, therefore, to multiply the proposed equation by this factor z, and it will become a complete differential.

414. Let the equation, for instance, be

$$ydx-xdy=0;$$

we obtain then

$$\frac{dM}{dy}-\frac{dN}{dx}=2,$$

which reduces the formula (109) to

$$\int\frac{dz}{z}=\int\frac{2dx}{-x};$$

whence we derive, by integrating,

$$\log z=-2\log x+\log C=-\log x^2+\log C=\log\frac{C}{x^2};$$

and passing to numbers, we find

$$z=\frac{C}{x^2};$$

the expression $\frac{C(ydx-xdy)}{x^2}$ will consequently be a complete differential.

415. We may find an infinite number of factors which possess the same property. For let z be a factor which renders the equation $Mzdx+Nzdy=0$ complete; representing the integral of this equation by u, we shall have

$$Mzdx+Nzdy=du;$$

multiplying the two sides by φu, we shall obtain

$$\varphi u(Mzdx+Nzdy)=\varphi udu;$$

and φu being arbitrary in its form, we may assume for it any function of u, for instance, $2u^2$, and then $2u^2du$ being a complete differential,

$$2u^2(Mzdx+Nzdy)=2u^2du$$

must be so also; so that the factor $2zu^2$ will have the property of rendering integrable the expression

$$Mdx+Ndy=0;$$

and we see that we may make an infinite number of similar hypotheses respecting φu.

Conditions of integrability of functions of three and a greater number of variables. Integration of equations of three variables which satisfy those conditions. The equation of condition necessary that the integration of differential equations between three variables may depend on a common factor, and the means of satisfying the proposed equation, when this equation of condition is not fulfilled.

416. Let it be proposed to determine the conditions of integrability of the differential of a function of three variables x, y, z.

This function being represented by u, we shall have

$$du=Mdx+Ndy+Pdz \quad . \; . \; . \; . \; (111),$$

where

$$M=\frac{du}{dx},\ N=\frac{du}{dy},\ P=\frac{du}{dz};$$

and these equations may be combined two and two together in the three following different ways:

$$1^{\circ}.\ \frac{du}{dx}=M,\ \frac{du}{dy}=N,$$

$$2^{\circ}.\ \frac{du}{dx}=M,\ \frac{du}{dz}=P,$$

$$3^{\circ}.\ \frac{du}{dy}=N,\ \frac{du}{dz}=P.$$

417. By a demonstration similar to the one which has been already given (art. 172), we shall, from these equations, deduce the following:

$$\frac{dM}{dy}=\frac{dN}{dx},\ \frac{dM}{dz}=\frac{dP}{dx},\ \frac{dN}{dz}=\frac{dP}{dy} \quad . \; . \; . \; (112);$$

and, generally, if there be n variables, we shall have as many equations of condition as these variables, taken two and two, can give distinct products, i. e. $\frac{n(n-1)}{2}$ equations of condition.

418. When the differential du is 0, the equation (111) is reduced to

$$Mdx+Ndy+Pdz=0\,;$$

and this may be put under the form

$$dz=mdx+ndy \quad . \; . \; . \; . \; (113),$$

by making

$$\frac{M}{P}=-m,\ \frac{N}{P}=-n \quad . \; . \; . \; . \; (114).$$

If now z be considered as a function of x and y, we may treat the equation (113) as though it contained only these two variables; and the condition of integrability will consequently be reduced to the one of art. 401; that is to say, the differential of m, taken in respect of y, and divided by dy, must be equal to the differential of n, taken in respect of x, and divided by dx. To obtain these expressions, we must observe that the first will not be simply $\frac{dm}{dy}$, but must have a second term arising from the differentiation of z, considered as a function of y, and which term, therefore, will be represented, art. 26, by $\frac{dm}{dz}\,\frac{dz}{dy}$. What has been said of the total differential, taken in respect of y, will apply also to the total differential, taken in respect of x, and the equation of condition (97), art. 401 will be, in the present case,

$$\frac{dm}{dy}+\frac{dm}{dz}\,\frac{dz}{dy}=\frac{dn}{dx}+\frac{dn}{dz}\,\frac{dz}{dx}\,;$$

whence, transposing and observing that, according to equation (113), $\frac{dz}{dx}=m$, and $\frac{dz}{dy}=n$, we have

$$\frac{dm}{dy}-\frac{dn}{dx}+n\frac{dm}{dz}-m\frac{dn}{dz}=0 \quad . \; . \; . \; . \; (115).$$

But by differentiating the equations (114), according to art. 16, we have

$$\frac{dm}{dy}=-\frac{P\frac{dM}{dy}-M\frac{dP}{dy}}{P^2},\quad \frac{dn}{dx}=-\frac{P\frac{dN}{dx}-N\frac{dP}{dx}}{P^2},$$

$$n\frac{dm}{dz}=\frac{N}{P}\,\frac{P\frac{dM}{dz}-M\frac{dP}{dz}}{P^2},\quad m\frac{dn}{dz}=\frac{M}{P}\cdot\frac{P\frac{dN}{dz}-N\frac{dP}{dz}}{P^2}\,;$$

and these values being substituted in the equation (115), the two last terms reduced, and the common denominator P^2 suppressed, we shall find, by changing all the signs,

$$P\frac{dM}{dy}-M\frac{dP}{dy}-P\frac{dN}{dx}+N\frac{dP}{dx}-N\frac{dM}{dz}+M\frac{dN}{dz}=0 \quad \ldots\ldots (116).$$

This, then, is the equation of condition necessary that z may be considered as a function of the two independent variables x and y; i. e. that there may be a determinate equation between the three variables; and if, consequently, we take at hazard an equation $Mdx+Ndy+Pdz=0$, between three variables, before knowing whether the equation (116) is satisfied, we are not at liberty to assume that one of the variables is a function of the other two; i. e. that the proposed differential equation necessarily infers the existence of some equation between x, y, and z; or, in other terms, that this differential equation has some single equation for its integral.

419. A differential equation between three variables, for which the equation (116) is not fulfilled, was for some time considered as absurd, or at least as unimportant; Monge, as we shall shortly show, proved that this idea was erroneous.

420. When the equations (112) are not satisfied, if we represent by λ the factor proper to render $Mdx+Ndy+Pdz$ a complete differential, the equations of condition (112) will become

$$\frac{d\lambda M}{dy}=\frac{d\lambda N}{dx},\quad \frac{d\lambda M}{dz}=\frac{d\lambda P}{dx},\quad \frac{d\lambda N}{dz}=\frac{d\lambda P}{dy};$$

and performing the differentiations, we obtain

$$\left.\begin{aligned}
M\frac{d\lambda}{dy}-N\frac{d\lambda}{dx}+\lambda\left(\frac{dM}{dy}-\frac{dN}{dx}\right)&=0\\
M\frac{d\lambda}{dz}-P\frac{d\lambda}{dx}+\lambda\left(\frac{dM}{dz}-\frac{dP}{dx}\right)&=0\\
N\frac{d\lambda}{dz}-P\frac{d\lambda}{dy}+\lambda\left(\frac{dN}{dz}-\frac{dP}{dy}\right)&=0
\end{aligned}\right\} \quad \ldots\ldots (117).$$

If now we multiply the first of these equations by P, the second by $-N$, and the third by M, and add, the terms without the brackets will destroy each other; and the equation being then divisible by λ, that factor will disappear, and there will remain

$$P\frac{dM}{dy}-P\frac{dN}{dx}-N\frac{dM}{dz}+N\frac{dP}{dx}+M\frac{dN}{dz}-M\frac{dP}{dy}=0;$$

a result the same with equation (116), and which agrees with what we have said at the end of the art. (418); for, in order that the equation proposed may become integrable by means of a factor λ, it is necessary that, as in all other sorts of integration, it should conduct us to a single equation between x, y, and z, a condition expressed by the equation (116). When this equation has been satisfied, the determination of the factor λ will depend on only two of the three equations of condition (117), since their combination with the equation (116) will produce the third *.

421. We will inquire now how we can determine the integral, when the equation (116) is satisfied; and for this purpose, considering one of the variables, z for instance, as constant, the proposed equation represented by

$$\mathrm{M}dx+\mathrm{N}dy+\mathrm{P}dz=0 \ \ldots\ldots \ (118),$$

will, on this hypothesis, necessarily reduce itself to

$$\mathrm{M}dx+\mathrm{N}dy=0 \ \ldots\ldots \ (119).$$

If this last equation be not immediately integrable, this may arise from some common factor having disappeared from the equation (118). Designating it by λ, and restoring it in the proposed equation, we shall have

$$\lambda\mathrm{M}dx+\lambda\mathrm{N}dy+\lambda\mathrm{P}dz=0 \ \ldots \ (120),$$

and making z constant, this equation will become

$$\lambda\mathrm{M}dx+\lambda\mathrm{N}dy=0 \ \ldots \ (121).$$

If, now, by any process, we find a factor which renders the equation

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* This is easily verified; for if we had, for example, the two equations

$$\mathrm{M}\frac{d\lambda}{dy}-\mathrm{N}\frac{d\lambda}{dx}+\lambda\left(\frac{d\mathrm{M}}{dy}-\frac{d\mathrm{N}}{dx}\right)=0,$$

$$\mathrm{N}\frac{d\lambda}{dz}-\mathrm{P}\frac{d\lambda}{dy}+\lambda\left(\frac{d\mathrm{N}}{dz}-\frac{d\mathrm{P}}{dy}\right)=0,$$

by adding the first multiplied by P to the second multiplied by M, and subtracting from this sum the product of the equation (116) by λ, we should find, by reducing and suppressing the common factor N,

$$\mathrm{M}\frac{d\lambda}{dz}-\mathrm{P}\frac{d\lambda}{dx}+\lambda\left(\frac{d\mathrm{M}}{dz}-\frac{d\mathrm{P}}{dx}\right)=0,$$

which is the second of the equations (117).

(119) integrable, we shall consider it as being what we have represented by λ; and the equation (121) becoming then a complete differential, we shall be able to obtain the integral, which we will express by V. This integral will be generally a function of the variables x, y, and of z, treated as constant; it will consequently be rendered complete by the addition of an arbitrary function of z, which we will designate by φz; so that we shall have

$$V+\varphi z=0 \;.\;.\;.\; (122);$$

and differentiating this equation in respect of z alone, we shall obtain

$$\left(\frac{dV}{dz}+\frac{d\varphi z}{dz}\right)dz.$$

But this quantity must be identical with the multiplier of dz in the equation (120), and consequently we shall have

$$\lambda P=\frac{dV}{dz}+\frac{d\varphi z}{dz} \;.\;.\;.\; (123);$$

whence we deduce

$$\frac{d\varphi z}{dz}=\lambda P-\frac{dV}{dz} \;.\;.\;.\; (124);$$

and since the function φz, which has been given by the integration, can contain no other variable than z, it will be the same with $\frac{d\varphi z}{dz}$; and by virtue, therefore, of the equation 124, $\lambda P-\frac{dV}{dz}$ must also reduce itself to a function of the variable z alone.

It follows, from what has preceded, that having ascertained that the equation (116) is satisfied, we must consider one of the variables, z for instance, as constant, which will reduce the equation (118) to the equation (119): we must examine then whether the two terms $Mdx+Ndy$ can become integrable by being multiplied by a quantity which we have designated by λ; and having arrived at this factor, we must determine V. The values of λ, $\frac{dV}{dz}$ and P being then substituted in the equation (124) will give us $\frac{d\varphi z}{dz}$, and consequently by integrating $\frac{d\varphi z}{dz}$, we shall obtain the value of φz; and this, along with the value of V, being substituted in the equation (122), will give us the integral required.

422. For example, let the equation proposed be

$$yzdx - xzdy + yxdz = 0 \quad . \; . \; . \quad (125).$$

This satisfies the equation (116), and we must therefore proceed first to integrate $yzdx - xzdy = 0$, considering z as constant; for which purpose, writing the equation thus,

$$z(ydx - xdy) = 0,$$

we observe that the part within the brackets becomes a complete differential when multiplied by $\frac{1}{y^2}$, and we recognize, therefore, that, in the present case, we have

$$\lambda = \frac{1}{y^2}, \text{ and } V = \frac{zx}{y}.$$

This last equation being differentiated in respect of z alone, the expression $\frac{dV}{dz}$ becomes $\frac{x}{y}$; and this value and that of λ being substituted in the equation (124), it becomes

$$\frac{d\varphi z}{dz} = \frac{P}{y^2} - \frac{x}{y},$$

and since P is no other than the multiplier of dz in the equation (125), restoring its value, we shall have

$$\frac{d\varphi z}{dz} = \frac{x}{y} - \frac{x}{y},$$

or

$$\frac{d\varphi z}{dz} = 0;$$

and therefore

$$\varphi z = constant.$$

This value and that of V convert the equation (122) into

$$\frac{zx}{y} + C = 0,$$

which is consequently the integral of the equation proposed.

423. As a second example, we will take the equation

$$zydx + xzdy + xydz + az^2dz = 0,$$

which equally satisfies the equation of condition (116). Integrating, therefore, $zydx+xzdy$, considering z as constant, we shall have

$$zxy+\varphi z=0 \quad . \; . \; . \quad (126);$$

and this integration having been effected without being under the necessity of restoring a factor, we see that, in the present case, λ may be considered as equal to unity. The expression $\frac{dV}{dz}$ will thus be obtained by simply differentiating the product zxy in respect of z, which will give $\frac{dV}{dz}=xy$; and by means of this value and that of P, which is $xy+az^2$, the equation (124) will become

$$\frac{d\varphi z}{dz}=xy+az^2-xy,$$

or

$$\frac{d\varphi z}{dz}=az^2;$$

whence, multiplying by dz, and integrating in respect of z, we shall obtain

$$\varphi z=\frac{az^3}{3}+C=0;$$

and we conclude, therefore, that the integral sought is

$$xyz+\frac{az^3}{3}+C.$$

424. When the equation (116) is not satisfied, it cannot be assumed that there exists an equation which, being differentiated, will produce the proposed equation; and, consequently, the equation (124), which rests on that hypothesis, can no longer subsist; as will appear evident in the following example: Let

$$zydx-zxdy+(z^2-y^2)dz=0$$

be an equation which does not satisfy the equation of condition (116). We will examine how, in this case, the part $\lambda P-\frac{dV}{dz}$, which would form the second side of the equation (124), did that equation hold good, is composed; and, for this purpose, considering z as constant, we shall have

$$zydx-zxdy=0,$$

an equation which becomes integrable when divided by xy; so that

$$\lambda=\frac{1}{xy},\ \mathrm{V}=x-z\log y\ ;$$

and consequently $\lambda\mathrm{P}-\frac{d\mathrm{V}}{dz}$ has for its value

$$\frac{z^2-y^2}{xy}-\log y.$$

Now this quantity being a function of the three variables x, y, z, cannot be reduced to a function of z alone, as the equation (124), did it hold good, would require; and therefore the equation (124) cannot, in the case before us, subsist.

425. Let now $\mathrm{M}dx+\mathrm{N}dy+\mathrm{P}dz=0$ be a differential equation, for which the equation of condition (116) is not fulfilled; designating by λ the factor proper to render only the part $\mathrm{M}dx+\mathrm{N}dy$ integrable, and multiplying the proposed equation by this factor, we shall have

$$\lambda\mathrm{M}dx+\lambda\mathrm{N}dy+\lambda\mathrm{P}dz=0\ .\ .\ .\ (127)\ ;$$

and integrating the part $\lambda\mathrm{M}dx+\lambda\mathrm{N}dy$, on the supposition of z being constant, the integral thus obtained may be represented, as in art. 421, by

$$\mathrm{V}+\varphi z=0.$$

If the differential of this equation be taken in respect of the three variables, we cannot thence assume it to be identical with the equation (127); for the equation of condition (116) not being fulfilled, it follows that the equation (127) cannot be considered as arising from the differentiation of some single equation; and since it is on this hypothesis that the equation (124) rests, we see that in this case it can subsist no longer; but though, when the equation (116) is not fulfilled, we are precluded from supposing that the equation proposed arises from the differentiation of some single equation, we may, however, change our hypothesis, and consider that equation as the result of some two equations. Let $\mathrm{V}+\varphi z=0$ be taken for the first; we may then assume for the second any arbitrary relation whatever betwixt x, y, z, provided always that, conjointly with the first, it destroy all the terms of the equation (127). Suppose, therefore, that this relation be the one given by the equation (124), an equation which could not subsist when it was required to satisfy the proposed equation, but which, on the present hypothesis, is admissible, since it is easy to see that, combined with the equation (122), it may satisfy the equation (127). For, differentiating the equation (122), in respect to the three variables, the equation (121) will furnish us first with the terms which arise from the differentiation taken in respect of x and y; since we have seen that the equation (122) was the integral of the equation (121), taken in respect of those two variables. Adding, then, to the terms $\lambda Mdx+\lambda Ndy$ thus obtained those which arise from the differentiation of the equation (122), taken in respect of z, we shall have

$$\lambda Mdx+\lambda Ndy+\frac{dV}{dz}dz+\frac{d\varphi z}{dz}dz=0;$$

and if, in this equation, we replace the two last terms by their values derived from the equation (124), we shall obtain

$$\lambda Mdx+\lambda Ndy+\lambda Pdz=0;$$

an equation in which we recognize the proposed one, and which, consequently, will be satisfied altogether by the two equations

$$V+\varphi z=0,\ \frac{dV}{dz}+\frac{d\varphi z}{dz}=\lambda P \quad . \ . \ . \ (128)$$

employed simultaneously.

426. Let us take, for example, the equation

$$ydy+zdx=dz;$$

if we consider z as constant, the factor proper to render the part $ydy+zdx$ integrable is 2, and consequently we shall have

$$2ydy+2zdx-2dz=0 \quad . \ . \ . \ . \ (129);$$

an equation which will be satisfied by the system of the two following equations

$$y^2+2zx+\varphi z=0,\ 2x+\frac{d\varphi z}{dz}+2=0 \quad . \ . \ . \ (130).$$

For the first being differentiated in respect to all the variables, will give

$$2ydy+2zdx+2xdz+\frac{d\varphi z}{dz}dz=0;$$

and deducing from this equation the value of $2ydy+2zdx$, and substituting it in the equation (129), that will become

$$-2xdz-\frac{d\varphi z}{dz}dz-2dz=0,$$

an equation which is also satisfied, by virtue of the second of the equations (130).

427. The equations (130) show us that the form of the function φz is altogether arbitrary, and that, consequently, if we make $\varphi z = z^3$, for instance, the original equation will be equally satisfied by the system of the two equations

$$y^2+2zx+z^3=0\,;\ 2x+3z^2+2=0 \quad . \ . \ . \ (131).$$

428. By means of these two equations between three variables, we may construct *(note ninth)* a curve of double curvature, which, at all its points, will satisfy the proposed equation; and if, instead of taking $\varphi z = z^3$, we should assume for φz some other function of z, we might determine another curve of double curvatures, which would equally satisfy the proposed equation; it follows, therefore, that the equations (130) represent a series of curves of double curvature, all of which satisfy the proposed equation, and are connected with each other by the common property that their equations differ from each other only by the terms represented by φz and $\frac{d\varphi z}{dz}$.

Theory of arbitrary constants.

429. An equation $V=0$ between x, y, and constants, may be considered as the complete integral of some differential equation, the order of which will depend on the number of constants which $V=0$ shall contain. These constants are termed arbitrary, because if one of them be represented by a, and V or one of its differentials be put under the form . . . . . $f(x,y)=a$, we see that a will be no other than the arbitrary constant introduced by the integration of $d\,.f(x,y)$.

This being premised, if the differential equation in question be of the order n, since each successive integration produces an arbitrary constant, it follows that $V=0$, which is supposed to be given us by these integrations, must contain at least n arbitrary constants more than our differential equation *.

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* If an equation between x and y should not contain n arbitrary constants more than the differential equation of the order n, it could not be considered as the primitive equation. For example, the equation $y=ax^3$, which by two

Let now

$$F(x,y)=0, F\left(x,y,\frac{dy}{dx}\right)=0, F\left(x,y,\frac{dy}{dx},\frac{d^2y}{dx^2}\right)=0 \ldots (132)$$

be the primitive of a differential equation of the second order, and its immediate derivatives; betwixt the two first of these three equations, we may eliminate successively the constants a and b, and so obtain

$$\varphi\left(x, y, \frac{dy}{dx}, b\right)=0, \varphi\left(x, y, \frac{dy}{dx}, a\right)=0 \ldots . (133).$$

If, without knowing $F(x, y)=0$, we had arrived at these equations, we should only have to eliminate $\frac{dy}{dx}$ between them in order to obtain $F(x, y)=0$, which would be the complete integral, since it contains the arbitrary constants a and b.

430. If, on the other hand, we eliminated these two constants betwixt the three equations (132), we should arrive at an equation, which, containing the same differential coefficients, might be represented by

$$\varphi\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right)=0 \ldots . (134);$$

but to this either of the equations (133) would also lead us. For the constant contained in one of these equations and its immediate differential being eliminated betwixt them, we should obtain separately two equations of the second order; and these could not differ from each other or from the equa-

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successive differentiations conducts us to $\frac{d^2y}{dx^2}=6ax$, is only a particular integral; and is obtained by making $b=0$ and $c=0$ in the complete integral, which is $y=ax^3+bx+c$.

It must be observed also that several constants attached to the same power of x are to be considered only as one; thus in the equation $y=(a+b)x+c$. we reckon $a+b$ but as one constant.

tion (134), or otherwise the values of x and y would not be the same in the one and the other. It follows, therefore, that a differential equation of the second order may arise from two differential equations of the first order, which are necessarily different, since the arbitrary constant in the one is not the same with the arbitrary constant in the other. The equations (133) are what we call the first integrals of the equation (134), which is unique, and the equation $F(x, y)=0$ is its second integral.

431. Let us take, for example, $y=ax+b$, which, on account of its two constants, may be considered as the primitive of an equation of the second order.

We deduce from it by differentiation, and the consequent elimination of a,

$$\frac{dy}{dx}=a,\ y=x\frac{dy}{dx}+b;$$

and these two first integrals of the equation of the second order which we are seeking, being each differentaited in turn, conduct equally, by the elimination of a and b, to the same equation $\frac{d^2y}{dx^2}=0$.

In the case in which the number of the constants is greater than that of the arbitrary constants required, the additional constants, being connected by the same equations, do not introduce any new relation. Let us investigate, for example, the equation of the second order, the primitive of which is

$$y=\tfrac{1}{2}ax^2+bx+c=0.$$

Differentiating this, we obtain

$$\frac{dy}{dx}=ax+b,$$

c and b being then eliminated successively between these equations, we have the two first integrals

$$\frac{dy}{dx}=ax+b,\ y=x\frac{dy}{dx}-\tfrac{1}{2}ax^2+c \quad . \ . \ . \ . \ (135);$$

and combining each of these with their immediate differentials, we arrive, by two different ways, at the same result $\frac{d^2y}{dx^2}=a$. If, on the other hand, we had eliminated the third constant a betwixt the primitive equation and its immediate differential, the result would have been in no way different; for we should have arrived first at the result that would be furnished us by the elimination of a betwixt the equations (135), and found then $x\frac{d^2y}{dx^2}=\frac{dy}{dx}-b$, an equation which we reduce to $\frac{d^2y}{dx^2}=a$, by combining it with the first of the equations (135).

432. Applying the same considerations to the differential equation of the third order, if we differentiate the equation $\mathrm{F}(x,y)=0$ three times in order, we shall have

$$\mathrm{F}\left(x,y,\frac{dy}{dx}\right)=0,\ \mathrm{F}\left(x,y,\frac{dy}{dx},\frac{d^2y}{dx^2}\right)$$
$$=0,\ \mathrm{F}\left(x,y,\frac{dy}{dx},\frac{d^2y}{dx^2},\frac{d^3y}{dx^3}\right)=0;$$

and these equations admitting the same values for each of the arbitrary constants which $\mathrm{F}(x,y)=0$ contains, we may in general eliminate these constants betwixt this last and the three preceding equations, and so obtain a result which we will represent by

$$f\left(x,y,\frac{dy}{dx},\frac{d^2y}{dx^2},\frac{d^3y}{dx^3}\right)=0 \quad . \ . \ . \ . \ (136).$$

This equation, from which the three arbitrary constants are eliminated, will be the differential equation of the third order of $\mathrm{F}(x,y)=0$; and conversely, $\mathrm{F}(x,y)=0$ will be the third integral of the equation (136).

433. If we eliminate each of the arbitrary constants succes-

sively between the equation $F(x, y)=0$ and the one immediately deduced from it by differentiation, we shall obtain three equations of the first order, which will be the second integrals of the equation (136).

Lastly, if we eliminate two of the three arbitrary constants, by means of the equation $F(x, y)=0$, and the equations deduced from it by two successive differentiations; i. e. if we eliminate the constants between the equations

$$F(x, y)=0,\ F\left(x, y, \frac{dy}{dx}\right)=0,\ F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right)=0 (137),$$

we shall retain, in the equations arising from the elimination, one of the three arbitrary constants successively; and consequently shall have as many equations as there are arbitrary constants.

Let a, b, c, be these arbitrary constants; the equations of which we speak, considered only in respect to the arbitrary constants they contain, may be represented thus:

$$\phi c=0,\ \varphi b=0,\ \varphi a=0\ .\ .\ .\ .\ (138);$$

and since the equations (137) contribute each of them to the elimination which gives us one of the last, it follows that the equations (138) will be each of the second order; they are called the first integrals of the equation (136).

434. Generally, a differential equation of an order n will have a number n of first integrals, which will consequently contain the differential coefficients from $\frac{dy}{dx}$ up to $\frac{d^{n-1}y}{dx^n}$ inclusively, i. e. a number $n-1$ of differential coefficients; and we see that when these equations are all known, we have only to eliminate the differential coefficients between them to obtain the primitive equation.

On the particular solutions of differential equations of the first order.

435. We have seen, art. 355, that a particular integral may

always be deduced from the complete integral, by giving a suitable value to the arbitrary constant contained in the latter. Suppose, for instance, that we have given the equation

$$xdx+ydy=dy\sqrt{x^2+y^2-a^2},$$

the complete integral of which is

$$y+c=\sqrt{x^2+y^2-a^2}:$$

if, for greater convenience of operating, we get quit of the roots, the proposed expression will become

$$(a^2-x^2)\frac{dy^2}{dx^2}+2xy\frac{dy}{dx}+x^2=0 \quad . \; . \; . \quad (139),$$

and we shall have for the complete integral

$$2cy+c^2-x^2+a^2=0 \quad . \; . \; . \quad (140);$$

when it is evident that by assuming for c a constant arbitrary value $c=2a$, we shall obtain the particular integral

$$2cy+5a^2-x^2=0,$$

which will possess the property of satisfying the proposed equation (139) equally well with the complete integral.

For we deduce from this particular integral

$$y=\frac{x^2-5a^2}{2c},\ \frac{dy}{dx}=\frac{x}{c};$$

by which values the proposed equation is reduced to

$$(x^2-a^2)\frac{x^2}{c^2}=\frac{x^2}{c^2}(x^2+c^2-5a^2),$$

and this becomes satisfied by substituting on the second side the value of c^2, which is furnished us by the relation $c=2a$, established between the constants.

For a long time it was supposed that this property of the complete integral was general, and that when a differential equation between x and y was given, we could not meet with a finite equation between the same variables, which was not a particular case of the complete integral, by giving, as we have just done, an arbitrary value to the constant; but it was at length discovered that this was not always the case, and Euler himself, in a memoir published in 1756, regarded as a paradox of the integral calculus the singular fact of the equation

$$x^2+y^2=a^2 \quad . \quad . \quad . \quad (141),$$

which possesses the property of satisfying the differential equation (139), and yet is not comprised in the complete integral. For the equation (141) being differentiated gives $xdx=-ydy$, and this value and that of x^2+y^2 being substituted in the equation (139) cause all the terms to disappear, and consequently satisfy the equation; nevertheless the equation (141) is not comprised in the complete integral; for whatever be the constant value we give to c in the equation (140), that equation can never lead to the equation (141), since the first, being that of a parabola, can never in any case become the equation (141), which is the equation of a circle.

This equation (141), which satisfies the one proposed without being contained in the complete integral, is called a particular or singular solution of the equation proposed. Clairault, about the year 1734, had remarked this fact, and it was for a long time supposed that equations of this sort were not connected with the complete integral; Lagrange showed that they were dependent on it, and on this subject laid down the theory which we shall proceed to develop.

436. Let $Mdx+Ndy=0$ be a differential equation of the first order of a function of two variables x and y; this equation may be conceived as arising from the elimination of some constant c betwixt an equation of the same order, which we will represent by $mdx+ndy=0$, and the complete integral $F(x, y, c)=0$, which we will designate by u. But since all

that is required is to take the constant c, so that the equation $Mdx+Ndy=0$ may be the result of the elimination, we see that, provided only the equation $Mdx+Ndy=0$ be satisfied, we may make the constant c itself vary; in which case the complete integral $F(x, y, c)=0$ will assume a still greater generality, and will represent an infinite number of curves of the same species, differing from each other only by a parameter, i. e. by a constant. This hypothesis is evidently admissible, since, when the equation $Mdx+Ndy=0$ is given, it is in the true spirit of analysis, not to reject any of the means by which this equation can be produced.

437. Suppose, therefore, that the complete integral being differentiated, considering c as variable, we have obtained

$$dy=\frac{dy}{dx}dx+\frac{dy}{dc}dc \quad . \; . \; . \quad (142).$$

For greater simplicity we will write this equation thus:

$$dy=pdx+qdc \quad . \; . \; . \quad (143);$$

and it is evident that if, whilst p continues finite, qdc become 0, the result of the elimination of c, considered as variable, between $F(x, y, c)=0$ and the equation (143) will be the same with that of the elimination of c, considered as constant, between $F(x, y, c)=0$ and the equation $dy=pdx$ *, for the equation (143), when qdc becomes 0, will not differ from $dy=pdx$. But that qdc may be $=0$, we must have one of the factors of this equation 0, i e. we must have

$$dc=0, \text{ or } q=0:$$

in the first of these cases, $dc=0$ gives $c=constant$, which is what takes place for a particular integral; and it will there-

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* This result is by hypothesis $Mdx+Ndy=0$.

fore be the second case only that can answer to a particular solution. But q being the coefficient of dc in the equation (142), we see that $q=0$ gives

$$\frac{dy}{dc}=0 \quad . \ . \ (143);$$

and this equation may contain c or be independent of it; if it contain c, two cases may happen: the equation $q=0$ will either contain c along with constants, or will contain c along with the variables. In the first case, the equation $q=0$ will still give $c=constant$; but in the second it will give . . . . . $c=f(x, y)$ *, and this value being substituted in the equation $F(x, y, c)=0$ will change it into another function of x and y, which will satisfy the equation proposed without being comprised in its complete integral, and will consequently be a particular solution: we shall, however, have only a particular integral if the equation $c=f(x, y)$, by means of the complete integral, be reduced to a constant.

438. When the factor $q=0$ of the equation $qdc=0$ does not contain the arbitrary constant c, we shall know whether the equation $q=0$ gives rise to a particular solution by combining it with the complete integral †. For example, if from $q=0$ we deduce $x=M$, and substitute this value in the complete integral $F(x, y, c)=0$, we shall obtain

$$c=constant=B, \text{ or } c=fy.$$

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* It being observed that this equation embraces, as particular cases, those in which we may have $c=fx$, or $c=fy$.

† In the latter case, in which q does not contain c, it may be asked how we have a right to equate q to zero. To this we shall answer, that the value given to c in the complete integral determines the equality of q to zero. For, when we deduce the value $x=fy$ from the equation $q=0$, to substitute it in $F(x, y, c)$, and obtain $F(y, fy, c)$, it is the same thing with deducing x from $F(x, y, c)=0$, and substituting its value in q; and consequently the result of this last operation will still be $F(y, fy, c)$. It only remains now to prove

In the first case, $q=0$ gives a particular integral; for, changing c into B in the complete integral, we shall merely be giving a particular value to the constant, just as we do when we pass from the complete integral to a particular one. In the second case, on the contrary, the value fy, introduced in place of c in the complete integral, will establish between x and y a relation different from what would result, were we to replace c merely by some constant arbitrary value; and in this case, therefore, we shall have a particular solution. What we have said of y, will apply in like manner to x.

439. It happens sometimes that the value of c presents itself under the form $\frac{0}{0}$: this indicates a factor common to the equations u and U which is foreign to them, and must be made to disappear. This results from a demonstration which, on account of its length, has been reserved for the notes *(note tenth)*.

440. We will now apply this theory to the investigation of particular solutions, when the complete integral is given.

Let the equation be

$$y\,dx-x\,dy=a\sqrt{dx^2+dy^2}\ \ldots\ (144),$$

the complete integral of which is determined in the manner following:

Dividing the equation by dx, and making $\frac{dy}{dx}=p$, we obtain first

$$y-px=a\sqrt{1+p^2}\ \ldots\ (145);$$

differentiating in respect of x, y, and p, we have

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that this result is equal to zero, to establish the same thing in respect to q; and this is done by considering $F(y, fy, c)$ as having arisen from the first operation, i. e. from $F(x, y, c)=0$, in which we have put for x its value.

$$dy-pdx-xdp=\frac{apdp}{\sqrt{1+p^2}};$$

and observing that $dy=pdx$, this equation is reduced to

$$xdp+\frac{apdp}{\sqrt{1+p^2}}=0;$$

which is satisfied by making $dp=0$.

This hypothesis gives us p = *constant* = c, a value which, being substituted in the equation (145), gives us

$$y-cx=a\sqrt{1+c^2} \;\ldots\; (146);$$

and this equation containing an arbitrary constant c, which does not appear in the proposed equation (144), it is consequently the complete integral.

441. This being premised, the part qdc of the equation (143) will be obtained by differentiating the equation (146), considering c as the only variable; an operation from which we shall have

$$xdc+\frac{acdc}{\sqrt{1+c^2}}=0;$$

and consequently the coefficient of dc, equated to zero, will give us

$$x=-\frac{ac}{\sqrt{1+c^2}} \;\ldots\; (147).$$

To disengage the value of c; raising this equation to the square, we shall find

$$(1+c^2)x^2=a^2c^2;$$

whence we shall deduce

$$c^2=\frac{x^2}{a^2-x^2},\ 1+c^2=\frac{a^2}{a^2-x^2},\ \sqrt{1+c^2}=\frac{a}{\sqrt{a^2-x^2}};$$

and, by means of this last equation, eliminating the surd quantity from the equation (147), we shall then obtain

$$c=-\frac{x}{\sqrt{a^2-x^2}} \quad \ldots\ldots (148)^*.$$

This value, and that of $\sqrt{1+c^2}$, being substituted in the equation (146), we shall have

$$y+\frac{x^2}{\sqrt{a^2-x^2}}=\frac{a^2}{\sqrt{a^2-x^2}};$$

whence we shall have

$$y=\frac{a^2-x^2}{\sqrt{a^2-x^2}}=\sqrt{a^2-x^2},$$

an equation which, being squared, will give us

$$y^2=a^2-x^2;$$

and we see that this equation is really a particular solution; for, by differentiating it, we obtain $dy=-\frac{xdx}{y}$; and this value and that of $\sqrt{x^2+y^2}$, being substituted in the equation (144), reduce it to $a^2=a^2$.

442. In the application which we have just given of the principles demonstrated, art. 437, we have determined the value of c by equating the differential coefficient $\frac{dy}{dc}$ to zero. This process will sometimes prove insufficient; for the equation

$$dy=pdx+qdc$$

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[*] We have not affected $\sqrt{1+c^2}$ with the double sign, because x and c being of contrary signs in the equation (147), the same must necessarily be the case in the equation (148).

being put under the form:

$$Adx+Bdy+Cdc=0,$$

where A, B, and C, are functions of x and y, we shall deduce from it

$$dy=-\frac{A}{B}dx-\frac{C}{B}dc\ \ldots\ (149),$$

$$dx=-\frac{B}{A}dy-\frac{C}{A}dc\ \ldots\ (150);$$

and we see that if all we have said of y, considered as a function of x, be applied to x, considered as a function of y, the value of the coefficient of dc will not necessarily result the same, since it is wanted only that some factor of B should destroy in C a factor different to what a factor of A could destroy in it, for the values of the coefficient of dc, on the two hypotheses, to result entirely different. Thus, though very generally the equations $\frac{C}{B}=0$, and $\frac{C}{A}=0$, give the same value for c, this does not always happen; and, on this account, when we have determined c by means of the equation $\frac{dy}{dc}=0$, it will not be altogether useless to see whether the hypothesis of $\frac{dx}{dc}=0$ produces the same result.

443. Clairaut first remarked a general class of equations susceptible of particular solutions: they are comprised under the form,

$$y=\frac{dy}{dx}x+F\left(\frac{dy}{dx}\right),$$

an equation which we may represent by

$$y=px+Fp\ \ldots\ .\ (151);$$

and differentiating, we shall find

$$dy = pdx + xdp + \frac{dFp}{dp}dp\,;$$

since $dy = pdx$, this equation is reduced to

$$xdp + \frac{dFp}{dp}dp = 0\,;$$

and dp being a common factor, it may be written thus:

$$\left(x + \frac{dFp}{dp}\right)dp = 0.$$

This equation will be satisfied by making $dp = 0$, which gives $p = \text{constant} = c$; and, consequently, substituting this value in the equation (151), we shall find

$$y = cx + Fc\,;$$

which equation will be the complete integral of the one proposed, since an arbitrary constant c has been introduced by the integration.

If we differentiate this equation in respect to c, we shall have

$$\left(x + \frac{dFc}{dc}\right)dc\,;$$

and, consequently, by equating the coefficient of dc to zero, we have the equation

$$x + \frac{dFc}{dc} = 0,$$

which, by the substitution of c in the complete integral, will give the particular solution (*note eleventh*).

Linear equations.

444. A differential equation between two variables x and y is linear, when the expressions $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3} \ldots \frac{d^ny}{dx_n}$, do not rise in the equation above the first degree; thus, supposing that A, B, C, D, ... N, X, are functions of x, the linear equation of the n^{th} order will be

$$Ay+B\frac{dy}{dx}+C\frac{d^2y}{dx^2}+D\frac{d^3y}{dx^3} \ldots . + N\frac{d^ny}{dx^n}=X \ldots . \quad (152).$$

445. When this equation is of the first order, it is reduced to

$$Ay+B\frac{dy}{dx}=X;$$

getting quit of the denominator, and dividing by B, we may put this under the form

$$dy+Pydx=Qdx,$$

and we have seen, art. 385, that this equation has for its integral

$$y=e^{-\int Pdx}[\int Qe^{\int Pdx}dx+C].$$

446. When the term X in the equation (152) is 0, if a number n of particular values, p, q, r, &c., substituted successively in place of y, have each the property of satisfying the equation, we have only to multiply p, q, r, &c., by the arbitrary constants, a, b, c, &c., to conclude that the finite complete integral is

$$y=ap+bq+cr+ \text{\&c.}$$

The demonstration of this proposition being the same for all orders, we shall consider only the equation

$$Ay+B\frac{dy}{dx}+C\frac{d^2y}{dx^2}+D\frac{d^3y}{dx^3}=0 \ldots . \quad (153);$$

in which substituting successively for y the hypothetical values p, q, r, &c., we shall have

$$Ap+B\frac{dp}{dx}+C\frac{d^2p}{dx^2}+D\frac{d^3p}{dx^3}=0,$$

$$Aq+B\frac{dq}{dx}+C\frac{d^2q}{dx^2}+D\frac{d^3q}{dx^3}=0,$$

$$Ar+B\frac{dr}{dx}+C\frac{d^2r}{dx^2}+D\frac{d^3r}{dx^3}=0\,;$$

and multiplying these three equations, the first by a, the second by b, and the third by c, and adding the results, we find

$$A(ap+bq+cr)+B\left(a\frac{dp}{dx}+b\frac{dq}{dx}+c\frac{dr}{dx}\right)$$

$$+C\left(a\frac{d^2p}{dx^2}+b\frac{d^2q}{dx^2}+c\frac{d^2r}{dx^2}\right)+D\left(a\frac{d^3p}{dx^3}+b\frac{d^3q}{dx^3}+c\cdot\frac{d^3r}{dx^3}\right)=0.$$

Now it is evident that this expression, which is identically 0, is the same that would have been obtained by making $y=ap+bq+cr$, in the equation (153); this value of y therefore satisfies the equation (153), and since it contains three arbitrary constants, it is the finite complete integral of that equation.

447. When X is not $=0$ in the equation

$$Ay+B\frac{dy}{dx}+C\frac{d^2y}{dx^2}+D\frac{d^3y}{dx^3}=X \quad . \; . \; . \; . \quad (154),$$

if we can find three particular values p, q, r, which, substituted successively for y, each of them satisfies the equation

$$Ay+B\frac{dy}{dx}+C\frac{d^2y}{dx^2}+D\frac{d^3y}{dx^3}=0 \quad . \; . \; . \; . \quad (155),$$

the finite complete integral of the equation (154) will be

$$y=ap+bq+cr \quad . \; . \; . \; . \quad (156)\,;$$

but in this case a, b, c, instead of being constants, will be functions of x, which we shall shortly see how to determine.

448. To demonstrate this theorem, we shall differentiate the equation (156), and divide by dx, when we shall have

$$\frac{dy}{dx}=a\frac{dp}{dx}+b\frac{dq}{dx}+c\frac{dr}{dx}+p\frac{da}{dx}+q\frac{db}{dx}+r\frac{dc}{dx}.$$

We shall now dispose of the three indeterminate quantities a, b, c, by three conditions; and by the first we shall make

$$p\frac{da}{dx}+q\frac{db}{dx}+p\frac{dc}{dx}=0 \quad \ldots\ldots \quad (157),$$

when there will remain

$$\frac{dy}{dx}=a\frac{dp}{dx}+b\frac{dq}{dx}+c\frac{dr}{dx};$$

and a new differentiation will give us

$$\frac{d^2y}{dx^2}=a\frac{d^2p}{dx^2}+b\frac{d^2q}{dx^2}+c\frac{d^2r}{dx^2}+\frac{da}{dx}\,\frac{dp}{dx}+\frac{db}{dx}\,\frac{dq}{dx}+\frac{dc}{dx}\,\frac{dr}{dx} \quad \ldots\ldots \quad (158).$$

For the second condition we shall assume

$$\frac{da}{dx}\,\frac{dp}{dx}+\frac{db}{dx}\,\frac{dq}{dx}+\frac{dc}{dx}\,\frac{dr}{dx}=0 \quad \ldots\ldots \quad (159),$$

whence there will remain

$$\frac{d^2y}{dx^2}=a\frac{d^2p}{dx^2}+b\frac{d^2q}{dx^2}+c\frac{d^2r}{dx^2};$$

and differentiating again and dividing by dx, there will result

$$\frac{d^3y}{dx^3}=a\frac{d^3p}{dx^3}+b\frac{d^3q}{dx^3}+c\frac{d^3r}{dx^3}+\frac{da}{dx}\,\frac{d^2p}{dx^2}+\frac{db}{dx}\,\frac{d^2q}{dx^2}+\frac{dc}{dx}\,\frac{d^2r}{dx^2}.$$

As a third condition, we shall suppose

$$\frac{da}{dx}\,\frac{d^2p}{dx^2}+\frac{db}{dx}\,\frac{d^2q}{dx^2}+\frac{dc}{dx}\,\frac{d^2r}{dx^2}=\frac{\mathrm{X}}{\mathrm{D}} \quad \ldots\ldots \quad (160),$$

and the preceding equation will become

$$\frac{d^3y}{dx^3}=a\frac{d^3p}{dx^3}+b\frac{d^3q}{dx^3}+c\frac{d^3r}{dx^3}+\frac{\mathrm{X}}{\mathrm{D}}.$$

We may say now that the value $y=ap+bq+cr$ satisfies the equation (154); for putting in this equation the value of y, and consequently those of its differential coefficients, which we have just determined, and effacing the terms in X, which destroy each other, we find

$$\left.\begin{array}{l}\mathrm{A}(ap+bq+cr)+\mathrm{B}\left(a\dfrac{dp}{dx}+b\dfrac{dq}{dx}+c\dfrac{dr}{dx}\right)\\ +\mathrm{C}\left(a\dfrac{d^2p}{dx^2}+b\dfrac{d^2q}{dx^2}+c\dfrac{d^2r}{dx^2}\right)+\mathrm{D}\left(a\dfrac{d^3p}{dx^3}+b\dfrac{d^3q}{dx^3}+c\dfrac{d^3r}{dx^3}\right)\end{array}\right\}=0 \quad \ldots \quad (161).$$

449. Since we do not yet know whether the value given to y makes all the terms of the equation (161) mutually destroy themselves, we must now proceed to demonstrate that this equation is identically 0. For this purpose, p, q, r, satisfying the equation (155), we have

$$Ap+B\frac{dp}{dx}+C\frac{d^2p}{dx^2}+D\frac{d^3p}{dx^3}=0,$$

$$Aq+B\frac{dq}{dx}+C\frac{d^2q}{dx^2}+D\frac{d^3q}{dx^3}=0,$$

$$Ar+B\frac{dr}{dx}+C\frac{d^2r}{dx^2}+D\frac{d^3r}{dx^3}=0;$$

and multiplying the first of these equations by a, the second by b, and the third by c, and adding the results, we shall find an equation identically 0, which will be the same with the equation (161).

450. To determine a, b, c, since the differential coefficients

$$\frac{da}{dx},\quad \frac{db}{dx},\quad \frac{dc}{dx},$$

enter into the equations of condition (157), (159), (160), only in the first degree, we may eliminate two of these coefficients, and so find the other in a function of the expressions $\frac{dp}{dx}$, $\frac{dq}{dx}$, &c., which are determinate functions of x; p, q, r, &c. being known; whence, therefore, we shall have equations of the form

$$\frac{da}{dx}=X_{,},\quad \frac{db}{dx}=X_{,,},\quad \frac{dc}{dx}=X_{,,,}$$

or,

$$da=X_{,}dx,\quad db=X_{,,}dx,\quad dc=X_{,,,}dx,$$

and integrating, we shall determine a, b, c.

451. This theorem is applicable to the case in which the linear equation is of any order whatever; and consequently the integration of such equations is reduced to that of the equation

$$Ay+B\frac{dy}{dx}+C\frac{d^2y}{dx^2}\ \ldots\ldots\ +N\frac{d^ny}{dx^n}=0\ \ldots\ldots\ (162).$$

452. When the linear equation of the order n has constant coefficients, it is easy to determine the integral. For if, in the equation (162), we make $y=e^{mx}$, we shall find, by differentiating,

$$\frac{dy}{dx}=e^{mx}m,\quad \frac{d^2y}{dx^2}=e^{mx}m^2,\quad \frac{d^3y}{dx^3}=e^{mx}m^3,\ \&c.;$$

and substituting these values in the equation (162), we shall obtain

$$e^{mx}(A+Bm+Cm^2 \dots +Nm^n)=0 \dots (163).$$

Let m', m'', m''', &c., be the roots of the equation

$$A+Bm+Cm^2 \dots +Nm^n=0 \dots (164),$$

the equation (162) will then be satisfied by the values

$$y=e^{m'x},\quad y=e^{m''x},\quad y=e^{m'''x},\ \&c.;$$

and since we have n values of y, the finite complete integral of the equation (162) will be

$$y=ae^{m'x}+be^{m''x}+ce^{m'''x}+\&c.$$

453. When $m'=m''$, the terms $ae^{m'x}$ and $be^{m''x}$ reduce themselves to $(a+b)e^{m'x}$; and since $a+b$ must then be considered as only one constant, we no longer have a number n of arbitrary constants in the expression for y. In this case, it is easily demonstrated that if $y=e^{m'x}$ satisfy the proposed equation, the value $y=xe^{m'x}$ must also satisfy it. For by differentiating this last equation, we find

$$\frac{dy}{dx}=xe^{m'x}m'+e^{m'x},\quad \frac{d^2y}{dx^2}=xe^{m'x}m'^2+2e^{m'x}m',$$

$$\frac{d^3y}{dx^3}=xe^{m'x}m'^3+3e^{m'x}m'^2,$$

$$\&c. = \&c.:$$

and these values reduce the equation (162) to

$$xe^{m'x}(A+Bm'+Cm'^2+Dm'^3+\&c.)$$
$$+e^{m'x}(B+2Cm'+3Dm'^2+\&c.) \dots (165).$$

But the equation (164) having, by hypothesis, two equal roots, we know, by the theory of equations, that the expression $B+2Cm+3Dm^2+\&c.$, will contain one root less than the proposed equation, and will vanish when we put $m=m'$, whence it follows that the expression (165) is identically 0. The equation (162) will consequently be satisfied by the value $y=xe^{m'x}$, and will have for its complete integral

$$y=ae^{m'x}+bxe^{m'x}+ce^{m''x}+\&c.$$

454. If there were three roots equal to m, we might prove, in like manner, that equation (162) would be satisfied by making

$$y=e^{m'x}+xe^{m'x}+x^2e^{m'x};$$

and so on.

455. When the equation (164) contains imaginary roots, if one of these roots be $h+k\sqrt{-1}$, the other will be $h-k\sqrt{-1}$, and we shall have, in the value of y, the two terms

$$ae^{hx+kx\sqrt{-1}}+be^{hx-kx\sqrt{-1}},$$

or

$$e^{hx}(ae^{kx\sqrt{-1}}+be^{-kx\sqrt{-1}})\ldots(166).$$

But we know that we have generally *(note fifth)* the formulæ

$$e^{\varphi\sqrt{-1}}=\cos\varphi+\sin\varphi\sqrt{-1},\quad e^{-\varphi\sqrt{-1}}=\cos\varphi-\sin\varphi\sqrt{-1};$$

the expression (166) therefore being compared with these formulæ, we may replace

$$e^{kx\sqrt{-1}} \text{ by } \cos kx+\sin kx\sqrt{-1},$$

$$e^{-kx\sqrt{-1}} \text{ by } \cos kx-\sin kx\sqrt{-1};$$

whence the formula (166) will become

$$e^{hx}(a\cos kx+a\sin kx\sqrt{-1}+b\cos kx-b\sin kx\sqrt{-1}),$$

an expression which may be written thus:

$$e^{hx}[(a+b)\cos kx+(a-b)\sin kx\sqrt{-1}]\cdots(167).$$

When X is 0, in the equation (152), a, b, c, being arbitrary constants, art. 446, we may suppose $a+b=c$, $a-b=c'\sqrt{-1}$, and then the imaginary part in the expression (167) will vanish.

On the integration of simultaneous equations.

456. Let it be proposed now to integrate at once two or more differential equations. Let

$$\left.\begin{aligned}My+Nx+P\frac{dy}{dt}+Q\frac{dx}{dt}&=T\\ M'y+N'x+P'\frac{dy}{dt}+Q'\frac{dx}{dt}&=T'\end{aligned}\right\}\ldots\ldots(168)$$

be the most general equations of the first degree between x, y, and the differential coefficients $\frac{dy}{dt}$, $\frac{dx}{dt}$; and in which the coefficients M, N, P, &c., are functions of the independent variable t.

These equations may be written thus:

$$(My+Nx)dt+Pdy+Qdx=Tdt,$$
$$(M'y+N'x)dt+P'dy+Q'dx=T'dt;$$

if we multiply the second of these by a function θ of t, and add the results, we shall obtain

$$[(M+M'\theta)y+(N+N'\theta)x]dt+(P+P'\theta)dy+(Q+Q'\theta)dx=(T+T'\theta)dt;$$

and representing the quantities within the parentheses by single letters, this equation may be written thus:

$$Hydt+Kxdt+Rdy+Sdx=Tdt;$$

whence we derive

$$H\left(y+\frac{K}{H}x\right)dt+R\left(dy+\frac{S}{R}dx\right)=Tdt \quad . \; . \; . \quad (169),$$

an equation which will be of the same form with the one

$$dy+Pydx=Qdx \quad . \; . \; . \quad (170),$$

which we have integrated, art. 385, if

$$d\left(y+\frac{K}{H}x\right)=dy+\frac{S}{R}dx \quad . \; . \; . \quad (171);$$

since then, by making

$$y+\frac{K}{H}x=z \quad . \; . \; . \quad (172),$$

the equation (169) will become

$$Hzdt+Rdz=Tdt,$$

or

$$dz+\frac{H}{R}zdt=\frac{T}{R}dt \quad . \; . \; . \quad (173);$$

and we see that this equation is of the same form with the equation (170), since $\frac{H}{R}$ and $\frac{T}{R}$ are certain functions of the independent variable t.

457. To satisfy the equation (171), we require only to have

$$d\left(\frac{K}{H}t\right)=\frac{S}{R}dt,$$

or, differentiating,

$$\frac{K}{H}dt+td.\frac{K}{H}=\frac{S}{R}dt\,;$$

and in order that this equation may be satisfied, the multipliers of dt must, in general, be equal, and consequently the term $td.\frac{K}{H}$ be 0; i. e. we must have

$$\frac{K}{H}=\frac{S}{R},\ d.\frac{K}{H}=0\ \ .\ .\ .\ (174).$$

Having substituted in these equations the values of the expressions . . . . . H, K, R, S, and performed the necessary differentiation, we must then eliminate θ contained in these equations, and we shall have the relation that must subsist between the coefficients, in order that the equation of condition may be satisfied.

458. In the case in which the coefficients of the first sides of the equations (168) are constants, the differential of a constant being equal to zero, there will remain only the first of the equations (174); this will suffice for determining the factor θ, which will then be constant, since it will become equal to a function of constants. Replacing K, H, R, S, by their values, we have

$$\frac{N+N'\theta}{M+M'\theta}=\frac{Q+Q'\theta}{P+P'\theta},$$

and getting quit of the denominators, we see that θ will be determined by an equation of the second degree, and therefore have two values.

Representing these by θ', θ'', and supposing that on substituting them successively in the equation (173), the coefficients of zdt and of dt become, in the first case, p' and q', in the second case p'' and q'', we shall have

$$dz+p'zdt=q'dt,$$
$$dz+p''zdt=q''dt\,;$$

whence, integrating according to the formula (90), art. (385), we shall find

$$z=e^{-\int p'dt}\left(\int q'e^{\int p'dt}dt\right)+C',$$
$$z=e^{-\int p''dt}\left(\int q''e^{\int p''dt}dt\right)+C''\,;$$

and substituting in these values that of z, deduced from the equation (172), we shall have two equations in x, y, and t.

459. If, excepting T, T', T'', which we shall always consider as functions of t, the coefficients M, N, P, Q, &c. be constants, and we have the three equations

$$\begin{aligned} dy+(\mathrm{M}y+\mathrm{N}x+\mathrm{P}z)dt&=\mathrm{T}dt,\\ dx+(\mathrm{M}'y+\mathrm{N}'x+\mathrm{P}'z)dt&=\mathrm{T}'dt,\\ dz+(\mathrm{M}''y+\mathrm{N}''x+\mathrm{P}''z)dt&=\mathrm{T}''dt; \end{aligned}$$

multiplying the second and third by constants θ, θ' respectively, and adding the results, we shall have an equation which may be represented by

$$dy+\theta dx+\theta' dz+\mathrm{Q}(y+\mathrm{R}x+\mathrm{S}z)dt=\mathrm{U}dt.$$

That this equation may be of the form

$$dy+\mathrm{P}ydx=\mathrm{Q}dx,$$

it is necessary that, considering $y+\mathrm{R}x+\mathrm{S}z$ as a single variable y', the differential dy' of this function should be equal to $dy+\theta dx+\theta' dz$, which requires us to have the equations of condition

$$\theta=\mathrm{R},\ \theta'=\mathrm{S};$$

and since R and S are functions only of θ and θ', by virtue of the preceding operations, it follows that these equations will suffice for determining the different values of the constants θ and θ'.

460. This method is general, and applies equally to differential equations of higher orders, since those equations can be reduced to the first degree. If we had, for example, the equations

$$\frac{d^2y}{dt^2}+\mathrm{M}y+\mathrm{N}x+\mathrm{P}\frac{dy}{dt}+\mathrm{Q}\frac{dx}{dt}=\mathrm{T},$$

$$\frac{d^2x}{dt^2}+\mathrm{M}'y+\mathrm{N}'x+\mathrm{P}'\frac{dy}{dt}+\mathrm{Q}'\frac{dx}{dt}=\mathrm{T}',$$

or rather

$$\left.\begin{aligned} d^2y+(\mathrm{M}y+\mathrm{N}x)dt^2+(\mathrm{P}dy+\mathrm{Q}dx)dt&=\mathrm{T}dt^2\\ d^2x+(\mathrm{M}'y+\mathrm{N}'x)dt^2+(\mathrm{P}'dy+\mathrm{Q}'dx)dt&=\mathrm{T}'dt^2 \end{aligned}\right\}\ \ldots\ (175),$$

we should make

$$dy=pdt,\ dx=qdt\ \ldots\ (176);$$

and observing that dt is constant, our equations would become

$$\begin{aligned} dp+(\mathrm{M}y+\mathrm{N}z+\mathrm{P}p+\mathrm{Q}q)dt&=\mathrm{T}dt,\\ dq+(\mathrm{M}'y+\mathrm{N}'z+\mathrm{P}'p+\mathrm{Q}'q)dt&=\mathrm{T}'dt: \end{aligned}$$

these two equations, with the equations (176), form four equations of the first degree, to which we may apply the preceding processes.

Integration of differential equations of the second order.

461. The general form for differential equations of the second order between two variables is

$$f\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right)=0 \ . \ . \ . \ (177).$$

We shall not attempt to integrate this equation in its utmost degree of generality; but shall proceed to examine how the integral can be found in certain particular cases.

462. We shall consider first the hypothesis on which we have

$$f\left(x, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right)=0 \ . \ . \ . \ . \ (178).$$

To integrate this equation we shall put $\frac{dy}{dx}=p$, $\frac{d^2y}{dx^2}=\frac{dp}{dx}$, when it will be reduced to

$$f\left(x, p, \frac{dp}{dx}\right) \ . \ \ . \ . \ (179);$$

and if this equation can be integrated, and we deduce from it $p=X$, we shall readily obtain the value of y, for the equation $\frac{dy}{dx}=p$ giving us $y=\int pdx$, if we substitute in this equation the value of p, we shall have $y=\int Xdx$. But if the equation (179), instead of giving us the value of p in terms of x, should give that of x in a function of p, so that we had $x=P$, integrating $dy=pdx$ by the method of parts, we should have

$$y=px-\int xdp,$$

and substituting in this equation the value of x, we should find

$$y=px-\int Pdp.$$

463. We will now consider the case in which we have

$$f\left(y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right)=0 \ldots (180):$$

making $\frac{dy}{dx}=p$, we shall find $\frac{d^2y}{dx^2}=\frac{dp}{dx}$; and replacing dx by its value $\frac{dy}{p}$, this equation will become

$$\frac{d^2y}{dx^2}=\frac{pdp}{dy}.$$

Putting these values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the equation (180), we shall transform it into

$$f(y, p, dy, dp)=0;$$

and if this equation give $p=\mathrm{Y}$, we must substitute this value in the equation $dx=\frac{dy}{p}$, when we shall obtain, by integrating,

$$x=\int\frac{dy}{\mathrm{Y}}.$$

If, on the contrary, y results as a function of p, and we have, consequently, $y=\mathrm{P}$; to obtain x, we must integrate the equation $dx=\frac{dy}{p}$ by parts, when we shall have

$$x=\frac{y}{p}+\int y\frac{dp}{p^2};$$

and substituting in this equation the value of y, we shall find

$$x=\frac{y}{p}+\int \mathrm{P}\frac{dp}{p^2}:$$

having integrated, we must then eliminate P by means of the equation $y=\mathrm{P}$.

464. When the equation (177) contains, along with $\frac{d^2y}{dx^2}$,

only one of the three quantities $\frac{dy}{dx}$, x and y, we have, in the first case,

$$f\left(\frac{dy}{dx}, \frac{d^2y}{dx^2}\right)=0 \; . \; . \; . \; (181);$$

and making $\frac{dy}{dx}=p$, and consequently $\frac{d^2y}{dx^2}=\frac{dp}{dx}$, and substituting these values in the equation (181), it will become

$$f\left(p, \frac{dp}{dx}\right)=0.$$

From this equation we deduce

$$\frac{dp}{dx}=\mathrm{P} \; . \; . \; . \; . \; (182),$$

and consequently

$$x=\int\frac{dp}{\mathrm{P}} \; . \; . \; . \; . \; (183).$$

On the other hand, the equation $\frac{dy}{dx}=p$ gives us

$$y=\int pdx;$$

and substituting in it the value of dx, given by the equation (182), we obtain

$$y=\int\frac{pdp}{\mathrm{P}} \; . \; . \; . \; . \; (184);$$

having integrated the equations (183) and (184), we must eliminate between them the quantity p, to obtain an equation between x and y.

465. In the case in which $\frac{d^2y}{dx^2}$ appears combined only with a function of x, we have

$$\frac{d^2y}{dx^2}=\mathrm{X};$$

multiplying by dx, and integrating, we find

$$\frac{dy}{dx}=\int Xdx+C;$$

representing $\int Xdx$ by X', we have

$$\frac{dy}{dx}=X'+C;$$

and multiplying anew by dx, and integrating, we obtain

$$y=\int X'dx+C.$$

466. Lastly, when $\frac{d^2y}{dx^2}$ is given in a function of y alone, we have only to integrate the equation

$$\frac{d^2y}{dx^2}=Y.$$

To accomplish this, we shall multiply the equation by $2dy$, which will give

$$2\frac{dy}{dx}.\frac{d^2y}{dx^2}=2Ydy;$$

and the first side being composed similarly to the differential of x^2, we shall find, by integrating,

$$\frac{dy^2}{dx^2}=\int 2Ydy+C;$$

extracting the square root of this, we shall obtain

$$\frac{dy}{dx}=\sqrt{\int 2Ydy+C},$$

and, by a new integration, we shall deduce

$$x=\int\frac{dy}{\sqrt{c+2\int Ydy}}.+C.$$

Partial differential equations of the first order.

467. An equation which subsists between differential coefficients, combined, according to the case, with variables and constants, is, generally, a partial differential equation, or, as it was formerly denominated, an equation with partial differences. These equations are thus denominated, because the notation of the differential coefficients which they contain indicates, as we have seen, art. 52, that the differentiation can only be effected partially, i. e., by considering certain of the variables as constants. This, consequently, supposes that the function proposed contains more than one variable. For greater simplicity, we will first admit only of two, and consider the partial differential equations of the first order, i. e., those which contain only one or more differential coefficients of the first order.

468. We will commence with integrating the following equation:

$$\frac{dz}{dx} = a.$$

If z, instead of being a function of two variables x and y, should contain only x, this would be no more than an ordinary differential equation, which, being integrated, would give . . . . $z = ax + c$; but since, in the present case, z is by hypothesis a function of x and y, the terms involving y in the function z must have disappeared by the differentiation, since in differentiating in respect of x, y has been considered constant. We must therefore, in integrating, adhere to the same hypothesis, and suppose that the arbitrary constant is in general a function of y; whence, consequently, we shall have for the integral of the equation proposed,

$$z = ax + \varphi y.$$

469. If we have also the partial differential equation

$$\frac{dz}{dx}=X,$$

in which X is a function of x; multiplying each side by dx, and integrating, we shall find

$$z=\int X dx+\varphi y.$$

470. For example, if the function represented by X should be x^2+a^2, the integral would be

$$z=\frac{x^3}{3}+a^2x+\varphi y.$$

471. We shall find no greater difficulty in integrating the equation

$$\frac{dz}{dx}=Y,$$

for which we shall have

$$z=Yx+\varphi y.$$

472. We may in like manner integrate every equation in which $\frac{dz}{dx}$ is equal to a function of two variables x and y.

If we have, for example,

$$\frac{dz}{dx}=\frac{x}{\sqrt{ay+x^2}},$$

considering y as constant, we shall multiply by dx, and integrate according to article (271); when representing by φy the constant which ought to be added to the integral, we shall have

$$z=\sqrt{ay+x^2}+\varphi y.$$

473. Lastly, if we have to integrate the equation

$$\frac{dz}{dx}=\frac{1}{\sqrt{y^2-x^2}},$$

y as usual being considered constant, we shall have (art. 274),

$$z = \sin^{-1}\cdot\frac{x}{y} + \varphi y.$$

474. Generally, to integrate the equation

$$\frac{dz}{dx}dx = \mathrm{F}(x, y)dx,$$

we must take the integral in respect of x, and adding then a constant function of y to complete it, we shall find

$$z = \int \mathrm{F}(x, y)dx + \varphi y.$$

475. From what has been said, we see that, excepting the hypothesis of one of the variables being constant, and the introduction, in the integral, of a constant function of that variable, we follow the same process as in the integration of ordinary differential equations.

476. Let us consider now the partial differential equations which contain two differential coefficients of the first order, and let the equation be

$$\mathrm{M}\frac{dz}{dx} + \mathrm{N}\frac{dz}{dy} = 0,$$

in which M and N represent given functions of x and y: we deduce from it

$$\frac{dz}{dy} = -\frac{\mathrm{M}}{\mathrm{N}}\frac{dz}{dx};$$

and substituting this value in the formula

$$dz = \frac{dz}{dx}dx + \frac{dz}{dy}dy \;.\;.\;.\;.\; (185),$$

which expresses nothing more than the condition that z is a function of x and y, we obtain

$$dz = \frac{dz}{dx}\left(dx - \frac{\mathrm{M}}{\mathrm{N}}dy\right).$$

or

$$dz=\frac{dz}{dx}\cdot\frac{\mathrm{N}dx-\mathrm{M}dy}{\mathrm{N}}.$$

Let λ be the factor proper to render $\mathrm{N}dx-\mathrm{M}dy$ a complete differential ds; we shall have then

$$\lambda(\mathrm{N}dx-\mathrm{M}dy)=ds \;\ldots\ldots\; (186),$$

and, by means of this equation, eliminating $\mathrm{N}dx-\mathrm{M}dy$ from the preceding one, we shall obtain

$$dz=\frac{1}{\lambda\mathrm{N}}\,\frac{dz}{dx}ds.$$

Lastly, observing that the value of $\frac{dz}{dx}$ is indeterminate, we may assume it such that $\frac{1}{\lambda\mathrm{N}}\cdot\frac{dz}{dx}ds$ shall be immediately integrable, which requires that $\frac{1}{\lambda\mathrm{N}}\frac{dz}{dx}$ be a function of s; for we know that the differential of every given function of s must be of the form $\mathrm{F}s\,.ds$. From this therefore it follows that we must have

$$\frac{1}{\lambda\mathrm{N}}\,\frac{dz}{dx}=\mathrm{F}s,$$

which will change the preceding equation into

$$dz=\mathrm{F}s ds;$$

whence we shall deduce

$$z=\phi s \;\ldots\ldots\; (187).$$

477. If we integrate by this method the equation

$$x\frac{dz}{dy}-y\frac{dz}{dx}=0 \;\ldots\ldots\; (188),$$

we have in this case $\mathrm{M}=-y$, $\mathrm{N}=x$, and the equation (186) will consequently become

$$ds=\lambda(xdx+ydy).$$

It is evident that the factor proper to render the second side of this equation integrable is 2; substituting therefore this value for λ, and integrating, we have

$$s=x^2+y^2;$$

whence, putting this value in the equation (187), we shall have for the integral of the equation (188)

$$z=\varphi(x^2+y^2).$$

478. Let now the equation be

$$P\frac{dz}{dx}+Q\frac{dz}{dy}+R=0 \quad .\ .\ .\ .\ (189),$$

in which P, Q, R, are functions of the variables x, y, z; dividing by P, and making $\frac{Q}{P}=M$, $\frac{R}{P}=N$, we may put it under the form

$$\frac{dz}{dx}+M\frac{dz}{dy}+N=0 \quad .\ .\ .\ .\ (190);$$

and making $\frac{dz}{dx}=p$, $\frac{dz}{dy}=q$, it will become

$$p+Mq+N=0 \quad .\ .\ .\ .\ (191).$$

This equation establishes a relation between the coefficients p and q in the general formula

$$dz=pdx+qdy \quad .\ .\ .\ .\ (192);$$

without this relation, p and q would be entirely arbitrary in the formula, since, as we have already observed, it does no more than indicate that z is a function of x and y, and that function may be any whatever. Thus in the equation (192) p and q must be considered as two indeterminate quantities; and eliminating p by means of the equation (191), we shall obtain

$$dz+\mathrm{N}dx=q(dy-\mathrm{M}dx) \quad . \ . \ . \quad (193);$$

in which q will still remain indeterminate: but we know (*see note third*) that when an equation of this sort holds good, whatever be the value of y, we must have separately

$$dz+\mathrm{N}dx=0,\ dy-\mathrm{M}dx=0 \quad . \ . \ . \quad (194).$$

479. If P, Q, and R do not contain the variable z, it will be the same with M and N; in which case the second of the equations (194) will be an equation between the two variables x and y, and may become a complete differential by means of a factor which we will represent by λ, when we shall have

$$\lambda(dy-\mathrm{M}dx)=0 \quad . \ . \ . \quad (195);$$

and the integral of this equation will be a function of x and y, to which we must add an arbitrary constant s, so that we shall have

$$\mathrm{F}(x,\ y)=s,$$

and consequently

$$y=f(x,\ s).$$

This value of y is the one given us by the second of the equations (194); and in order, therefore, that the two may hold good simultaneously, this value of y must be substituted in the first of the equations (194); for though the variable y does not explicitly show itself in that equation, we see that it may be contained in N.

This substitution, from the nature of the value which we have just found for y, comes to the same thing with considering y, in the first of the equations (194), as a function of x and s; and the first equation being, therefore, integrated on such hypothesis, we shall find

$$z=-\int \mathrm{N}dx+\varphi s.$$

480. To give an example of this mode of integration, let us

take the equation

$$x\frac{dz}{dx}+y\frac{dz}{dy}=a\sqrt{x^2+y^2}\,;$$

comparing it with the equation (190), we have

$$\mathrm{M}=\frac{y}{x},\ \mathrm{N}=-a\frac{\sqrt{x^2+y^2}}{x}\quad\ldots\quad(196),$$

and these values being substituted in the equations (194), they will be changed into

$$dz-a\frac{\sqrt{x^2+y^2}}{x}dx=0,\ dy-\frac{y}{x}dx=0\quad\ldots\quad(197).$$

Let λ be the factor which renders this last equation integrable; we shall have, then,

$$\lambda\left(dy-\frac{y}{x}dx\right)=0,$$

or

$$\lambda\left(\frac{xdy-ydx}{x}\right)=0,$$

and this equation will be integrable, if we make $\lambda=\frac{1}{x}$; since in that case its first side becomes a complete differential (art. 403).

Equating, therefore, the integral of this equation to an arbitrary constant s, we shall have

$$\frac{y}{x}=s,$$

and consequently

$$y=sx.$$

By means of this value of y, the first of the equations (197) is changed into

$$dz - a\frac{\sqrt{x^2+s^2x^2}}{x}dx = 0,$$

or

$$dz = adx\sqrt{1+s^2};$$

whence, integrating and considering s as constant, we shall obtain

$$z = a\int dx\sqrt{1+s^2} + \varphi s,$$

and consequently

$$z = ax\sqrt{1+s^2} + \phi s.$$

Replacing the value of s, there results lastly

$$z = ax\sqrt{1+\frac{y^2}{x^2}} + \varphi\frac{y}{x},$$

or

$$z = a\sqrt{x^2+y^2} + \varphi\frac{y}{x}.$$

481. In the most general case, in which the coefficients P, Q, R, of the equation (189) contain the three variables x, y, z, it may happen that the equations (194) contain each of them only the two variables which explicitly show themselves in the respective equations; and that consequently we may put them under the form

$$dz = f(x, z)dx, \quad dy = F(x, y)dx.$$

We cannot integrate these equations independently of each other, by supposing, as in art. 474,

$$z = \int f(x, z)dx + \varphi z, \quad y = \int F(x, y)dx + \Phi y;$$

for in this case we see that we must assume z to be constant in the first equation, and y to be constant in the second; hypotheses which are contradictory to each other, since one of the three coordinates x, y, z, cannot be supposed constant in the first equation without its being also constant in the second.

482. The following is the method by which we must integrate the equations (194), in the case in which they contain each of them only the two variables which expressly appear: let μ and λ be the factors which render the equations (194) complete differentials; representing these differentials by $d\mathrm{U}$ and $d\mathrm{V}$, we shall have

$$\lambda(dz+\mathrm{N}dx)=d\mathrm{U},\ \mu(dy-\mathrm{M}dx)=d\mathrm{V},$$

and by means of these values, the equation (193) will become

$$d\mathrm{U}=q\frac{\lambda}{\mu}d\mathrm{V}\quad .\ .\ .\ (198).$$

Since the first side of this equation is a complete differential, the second must be so likewise, which requires that $q\frac{\lambda}{\mu}$ be a function of V; representing this function by $\varphi\mathrm{V}$, the equation (198) will become

$$d\mathrm{U}=\varphi\mathrm{V}d\mathrm{V};$$

whence we shall deduce, by integrating,

$$\mathrm{U}=\Phi\mathrm{V}.$$

483. Let us take, for example, the equation

$$xy\frac{dz}{dx}+x^2\frac{dz}{dy}=yz:$$

this being written thus,

$$\frac{dz}{dx}+\frac{x}{y}\frac{dz}{dy}-\frac{z}{x}=0,$$

and compared with the equation (190), we shall have

$$M=\frac{x}{y},\ N=-\frac{z}{x},$$

by means of which values the equations (194) will become

$$dz-\frac{z}{x}dx=0,\ dy-\frac{x}{y}dx=0,$$

and getting quit of the denominators, we shall have

$$xdz-zdx=0,\ ydy-xdx=0.$$

The factors proper to render these equations integrable are $\frac{1}{x^2}$ and 2; substituting these, and integrating, we find $\frac{z}{x}$ and y^2-x^2 for the integrals; and putting these values in place of U and V, in the equation $U=\Phi V$, we shall obtain, for the integral of the equation proposed,

$$\frac{z}{x}=\Phi(y^2-x^2).$$

484. It is to be observed that if we had eliminated q instead of p, the equations (194) would have been replaced by the following ones:

$$Mdz+Ndy=0,\ dy-Mdx=0\ \ .\ .\ .\ (199);$$

and since all that we have said of the equations (194) will apply equally to these, it follows that, in the case in which the first of the equations (194) is not integrable, we are at liberty to replace those equations by the system of the equations (199); i. e. to employ the first of the equations (199) in place of the first of the equations (194), and then see if the integration be possible.

485. For example, if we had

$$az\frac{dz}{dx}-zx\frac{dz}{dy}+xy=0,$$

this equation, divided by az and compared with the equation (190), would give us

$$M=-\frac{x}{a},\ N=\frac{xy}{az};$$

whence the equations (194) would become

$$dz+\frac{xy}{az}dx=0,\ dy+\frac{x}{a}dx=0;$$

and getting quit of the denominators, we should have

$$azdz+xydx=0,\ ady+xdx=0\ \ .\ \ .\ \ .\ \ (200).$$

Now the first of these equations, containing three variables, cannot be immediately integrated; we shall therefore replace it by the first of the equations (199), when we shall have, in lieu of the equations (200), the following ones:

$$-\frac{x}{a}dz+\frac{xy}{az}dy=0,\ ady+xdx=0;$$

suppressing $\frac{x}{a}$ as a common factor in the first of these equations, and multiplying the one by $2z$ and the other by 2, we shall find

$$-2zdz+2ydy=0,\ 2ady+2xdx=0,$$

equations which have for their integrals

$$y^2-z^2, \text{ and } 2ay+x^2;$$

and substituting these values in place of U and V, we shall have

$$y^2-z^2=\varphi(2ay+x^2).$$

486. It may be observed that the first of the equations (199) is no other than the result of the elimination of dx between the equations (194).

Generally, we may eliminate every variable contained in the coefficients M and N; and, in a word, combine the equations (194) in any manner whatever; if, after having performed these operations, we obtain two integrals represented by $U=a$ and $V=b$, a and b being two arbitrary constants, we may always conclude, thence, that the integral is . . . . $U=\Phi V$. For, since a and b are two arbitrary constants, having taken b at pleasure, we may form a of b in any manner whatever; i. e. we may assume a as an arbitrary function of b; a condition which will be expressed by the equation . . . . $a=\varphi b$ (*note twelfth*); we shall consequently have the equations $U=\varphi b$, $V=b$, in which x, y, and z represent the same coordinates; and if we eliminate b between these equations, we shall obtain $U=\varphi V$. It may also be observed that this equation informs us that on making $V=b$, we ought to have $U=\varphi b=$ *constant*; i. e. that U and V are constants at the same time, without a and b being dependent one on the other, since the function φ is arbitrary. Now this is precisely the condition which is given us by the equations $U=a$ and $V=b$.

487. To give an application of this theorem, let

$$zx\frac{dz}{dx}-zy\frac{dz}{dy}-y^2=0.$$

Having divided by zx, we shall compare this equation with the equation (190), which will give us

$$M=-\frac{y}{x},\ N=-\frac{y^2}{xz},$$

and the equations (194) will become

$$dz-\frac{y^2}{xz}dx=0,\ dy+\frac{y}{x}dx=0,$$

or

$$zxdz-y^2dx=0,\ xdy+ydx=0.$$

The first of these equations containing three variables, we shall not attempt to integrate it in this state; but if we substitute the value of ydx, derived from the second, it will acquire a common factor x, which being suppressed, it is reduced to

$$zdz+ydy=0,$$

and we see that on multiplying it by 2, it becomes integrable; and the other equation being so likewise, we shall find, by integrating them,

$$z^2+y^2=a,\ xy=b;$$

whence we conclude that

$$z^2+y^2=\varphi xy.$$

488. We shall terminate what we have to say respecting partial differential equations of the first order, by the solution of this problem: *An equation which contains an arbitrary function of one or more variables being given, to find the partial differential equation which has produced it.*

Suppose, therefore that we have

$$z=\mathrm{F}(x^2+y^2);$$

we shall put

$$x^2+y^2=u\ .\ .\ .\ (201),$$

when our equation will become

$$z=\mathrm{F}u;$$

and since the differential of u must, in general, be a function of u, multiplied by du, we may assume

$$dz=\varphi udu.$$

If now we take the differential of z, in respect of x only, i. e. considering y as constant, we must take the differential of u

also on the same hypothesis; and, consequently, dividing the preceding equation by dx, we shall have

$$\frac{dz}{dx}=\varphi u\frac{du}{dx} \quad . \; . \; . \quad (202);$$

considering, then, x as constant, and y as variable, we shall find, by a similar process,

$$\frac{dz}{dy}=\varphi u\frac{du}{dy} \quad . \; . \; . \quad (203).$$

The values of the differential coefficients $\frac{du}{dx}$ and $\frac{du}{dy}$ which enter into the equations (202) and (203) will be obtained by differentiating the equation (201), in respect to x and y successively, which will give us

$$\frac{du}{dx}=2x,\ \frac{du}{dy}=2y;$$

substituting these values in the equations (202) and (203), we shall have

$$\frac{dz}{dx}=2x\varphi u,\ \frac{dz}{dy}=2y\varphi u;$$

and eliminating φu between these equations, we shall find, lastly,

$$y\frac{dz}{dx}=x\frac{dz}{dy}.$$

489. Let us take also, for example, the equation

$$z^2+2ax=\mathrm{F}(x-y).$$

Making

$$x-y=u \quad . \; . \; . \; . \quad (204),$$

this equation becomes

$$z^2+2ax=\mathrm{F}u;$$

and differentiating, we have

$$d(z^2+2ax)=\varphi u du\ ;$$

taking the differential indicated, in respect to x, we must consider z as variable, by virtue of x which is contained in it, and dividing by dx, we shall have

$$2z\frac{dz}{dx}+2a=\varphi u\frac{du}{dx}\quad .\ .\ .\ (205)\ ;$$

proceeding in a similar manner for y, considering z as a function which varies only on account of y, and dividing by dy, we shall find

$$2z\frac{dz}{dy}=\varphi u\frac{du}{dy}\quad .\ .\ .\ (206).$$

To eliminate the differential coefficients of du, the equation (204) gives us

$$\frac{du}{dx}=1,\ \frac{du}{dy}=-1\ ;$$

substituting these values in the equations (205) and (206), we shall have

$$2z\frac{dz}{dx}+2a=\varphi u,\ 2z\frac{dz}{dy}=-\varphi u\ ;$$

and eliminating φu between these equations, we shall obtain, lastly,

$$\frac{dz}{dx}+\frac{dz}{dy}+\frac{a}{z}=0.$$

Partial differential equations of the second order.

490. A partial differential equation of the second order, in which z is a function of two variables x and y, must always contain one or more of the differential coefficients

$$\frac{d^2z}{dx^2},\quad \frac{d^2z}{dy^2},\quad \frac{d^2z}{dxdy},$$

independently of the differential coefficients of the first order which it may contain.

491. We shall confine ourselves to integrating the most simple of the partial differential equations of the second order, and shall commence with the one:

$$\frac{d^2z}{dx^2}=0;$$

multiplying this by dx, and integrating in respect of x, we must add to the integral an arbitrary function of y, when we shall have

$$\frac{dz}{dx}=\varphi y;$$

multiplying anew by dx, and designating by ψy the function of y to be added to the integral, we shall find

$$z=x\varphi y+\psi y.$$

492. Let it be proposed now to integrate the equation

$$\frac{d^2z}{dx^2}=\mathrm{P},$$

in which P is a function of x and y; proceeding as before, we shall find first

$$\frac{dz}{dx}=\int\mathrm{P}dx+\varphi y;$$

and a second integration will give us

$$z=\int\left[\int\mathrm{P}dx+\varphi y\right]dx+\psi y.$$

493. We might integrate in the same manner

$$\frac{d^2z}{dy^2}=\mathrm{P}.$$

and we should find

$$z=\int\left[\int \mathrm{P}dy+\varphi x\right]dy+\psi x.$$

494. The equation

$$\frac{d^2z}{dydx}=\mathrm{P}$$

must be integrated, first in respect to one of the variables, and then in respect to the other, which will give

$$z=\int\left[\int \mathrm{P}dx+\varphi y\right]dy+\varphi x.$$

495. Generally, we shall treat in a similar manner any one of the equations

$$\frac{d^nz}{dy^n}=\mathrm{P},\ \frac{d^nz}{dxdy^{n-1}}=\mathrm{Q},\ \frac{d^nz}{dx^2dy^{n-2}}=\mathrm{R},\ \&c.,$$

in which P, Q, R, &c., are functions of x and y; and this will lead to a series of integrations, each of which will introduce an arbitrary function into the integral.

496. Among the equations now under consideration, one of the most easy of integration is the following:

$$\frac{d^2z}{dy^2}+\mathrm{P}\frac{dz}{dy}=\mathrm{Q}\,;$$

by P and Q designating always two functions of x and y. Making

$$\frac{dz}{dy}=u \quad .\ .\ .\ .\ (207),$$

we shall change this equation into

$$\frac{du}{dy}+\mathrm{P}u=\mathrm{Q} \quad .\ .\ .\ .\ (208).$$

To integrate, we shall consider x as constant, and then this equation will contain only two variables y and u, and will be of the same form with the equation

$$dy+Pydx=Qdx \quad \ldots\ldots (209),$$

treated of, art. 385, and the integral of which is

$$y=e^{-\int Pdx}\left(\int Qe^{\int Pdx}dx+C\right) \quad \ldots (210):$$

comparing therefore the equations (208) and (209), we shall have

$$y=u,\ x=y\,;$$

substituting these values in the formula (210), and changing C into φx, we shall obtain

$$u=e^{-\int Pdy}\left(\int Qe^{\int Pdy}dy+\varphi x\right);$$

and putting this value of u in the equation (207), multiplying by dy, and integrating, we shall find

$$z=\int\left[e^{-\int Pdy}\left(\int Qe^{\int Pdy}dy+\varphi x\right)\right]dy+\psi x.$$

497. We might integrate by the same method the equations

$$\frac{d^2z}{dxdy}+P\frac{dz}{dx}=Q,\ \frac{d^2z}{dxdy}+P\frac{dz}{dy}=Q,$$

in which P and Q represent functions of x, and by reason of the divisor $dxdy$, we see that the value of z cannot contain arbitrary functions of the same variable.

On the determination of the arbitrary functions which enter into the integrals of partial differential equations of the first order.

498. The arbitrary functions which complete the integrals of partial differential equations must be determined by the conditions which belong to the nature of the problems which have produced those equations, problems which for the most part belong to questions in physical mathematics. Not to wander too far from our subject, we shall confine ourselves to

considerations purely analytical, and investigate first what are the conditions contained in the equation

$$\frac{dz}{dx}=a \quad . \; . \; . \; . \quad (211).$$

499. So long as z is a function of x and y, this equation may be considered as that of a surface; and this surface, from the nature of its equation, will possess this property, that $\frac{dz}{dx}$ must always be a constant quantity. It follows from this that every section EF (fig. 87) of this surface, made by a plane CD Fig. 87. parallel to that of x, z, is a straight line. For, whatever be the nature of this section, if we divide it into an infinite number of parts mm', $m'm''$, $m''m'''$, &c., these parts, seeing that they are exceedingly small in length, may be considered as straight lines, and will represent the elements of the section; also any one of these elements mm' makes with a parallel mn to the axis of the abscissæ, an angle the trigonometrical tangent of which is represented by $\frac{dz}{dx}$; and since this angle is constant (because $\frac{dz}{dx}$ is so), it follows that all the angles $m'mn$, $m''m'n'$, $m'''m''n''$, &c., formed by the elements of the curve with the parallels mn, $m'n'$, $m''n''$, &c., to the axis of the abscissæ, will be equal; which proves that the section EF is a straight line.

500. We might have arrived at the same result by considering the integral of the equation $\frac{dz}{dx}=a$, which we have seen to be, art. 468,

$$z=ax+\varphi y \quad . \; . \; . \; . \quad (212);$$

for, for all the points of the surface which are in the plane CD, the ordinate is equal to a constant c, represented in the fig. 87, by AB; replacing therefore φy by φc, and making $\varphi c=C$, the equation (212) will become

$$z=ax+C \quad . \; . \; . \; . \quad (213),$$

and this equation being that of a straight line, the section EF, to which it belongs, must consequently be a straight line.

501. The same being the case in respect to the other secant planes which might be drawn parallel to that of $x.z$, we conclude that the sections of all these planes with the surface will be straight lines, which will be parallel to each other, since they will form each of them with a parallel to the axis of x, an angle whose trigonometrical tangent will be the constant a.

502. If now we make $x=0$, the equation (212) will be reduced to $z=\varphi y$, and will be that of a curve GHK traced along the plane of y, z; this curve comprising all the points of the surface, for which $x=0$, it will meet the plane CD in a point m (fig. 87), which will have, for one of its coordinates, $x=0$; and since we have also $y=AB=c$, the third coordinate, by virtue of the equation (213), will be $z=C$, a value represented in the figure by Bm. What we have said of the plane CD will apply to all the other planes, which are parallel to it, and it follows therefore that through all the points of the curve whose equation is $z=\varphi y$, and which is traced along the plane of y, z, straight lines will pass parallel to the axis of x.

Here then is all of which the equations (211) and (212) inform us, and since this condition is always fulfilled, whatever be the figure of the curve of which $z=\varphi y$ is the equation, we see that this curve is arbitrary.

503. It follows from what has been said, that the curve GHK, of which $z=\varphi y$ is the equation, may be composed of arcs of different curves, which join at their extremities*, as in

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* In this case, the curve will be determined by means of several equations, in such a manner that the first will give the value of the variable x, for instance, from $x=a$ to $x=b$; the second will give it from $x=b$ to $x=c$, and so on.

fig. 88, or of curves which leave breaks in certain places, as in Fig. 88. fig. 89. In the first case, the curve is discontinuous, in the Fig. 89. second it is discontiguous: in this last case it may be observed that two different ordinates PM and PN (fig. 89) correspond to the same abscissa AP. It is possible, lastly, that without being discontiguous, the curve may be composed of an infinite series of indefinitely small arcs, which belong each to different curves; in this case the curve is irregular, as, for example, the fibres of a feather would be which we should draw out at random; but however the curve may be formed of which the equation is $z = \varphi y$, it will suffice for constructing the surface to make a straight line move always parallel to itself, with the condition that its point M shall run along the curve GHK, of which the equation is $z = \varphi y$, and which is traced at random along the plane of y, z.

504. If, instead of the equation $\frac{dz}{dx} = a$, we had the one . . . . $\frac{dz}{dx} = X$, in which X was a function of x; then drawing a plane CD (fig. 87) parallel to that of x, z, the surface would be cut in a section EF, which would no longer be a straight line, as in the preceding case. For, for any point m', taken in this section, the trigonometrical tangent of the angle $n'm'm''$ formed by the prolongation of the element $m'm''$ of the section, with a parallel to the axis of x, will be equal to a function X of the abscissa x at that point; and since the abscissa x is different for every point, it follows that the angle $n'm'm''$ will be different for each point of the section, which shows us that EF will not, as before, be a straight line. The surface will be constructed in the same manner as in the preceding problem, by making the section EF move parallel to itself, so that its point m shall continually be in the curve GHK, of which the equation is $z = \varphi y$.

505. Suppose now that in the preceding equation, instead

of X we had a function P of x and y, the equation $\frac{dz}{dx} = \mathrm{P}$, containing three variables, will belong still to a curve surface. If we cut this surface by a plane parallel to that of x, z, we shall have a section in which y will be constant; and since in all its points, $\frac{dz}{dx}$ will be equal to a function of the variable x, it will follow, as in the preceding case, that this section will be a curve.

The equation $\frac{dz}{dx} = \mathrm{P}$, being integrated, we shall have, for that of the surface

$$z = \int \mathrm{P}dx + \varphi y\,;$$

if in this equation we give successively to y the increasing values y', y'', y''', y^{iv}, &c., and represent by P', P'', P''', P^{iv}, &c. what the function P then becomes, we shall have the equations

$$\left.\begin{array}{ll} z = \int \mathrm{P}'dx + \varphi y', & z = \int \mathrm{P}''dx + \varphi y'', \\ z = \int \mathrm{P}'''dx + \varphi y''', & z = \int \mathrm{P}^{\text{iv}}dx + \varphi y^{\text{iv}}, \\ \&\text{c.} = \&\text{c.} & \&\text{c.} = \&\text{c.} \end{array}\right\} \quad . \;.\;.\; (214)\,;$$

and we see that these equations will belong to curves of the same nature, but different in form, since the values of the constant y are not the same in all. These curves will be no other than the sections of the surface by planes parallel to that of x, z; and, in meeting the plane of y, z, they will form a curve, the equation of which will be obtained by making $x = 0$ in the equation of the surface. Representing by Y, what $\int \mathrm{P}dx$ becomes in this case, we shall have

$$z = \mathrm{Y} + \varphi y \quad . \;.\;.\; (215)\,;$$

and we see that on account of φy, the curve determined by this equation must be arbitrary; thus, having traced along the plane of y, z, the curve QRS (fig. 90) at pleasure, if we represent by RL the section of which $z=\int P'dx+\varphi y'$ is the equation, we must have this section to move, keeping its extremity always in the curve QRS, and in such a manner that, in its course, this section RL shall assume the successive forms determined by the equations (214); when we shall construct the surface to which the equation $\frac{dz}{dx}=P$ belongs. Fig. 90.

506. Let us consider, lastly, the general equation

$$\frac{dz}{dx}+M\frac{dz}{dy}+N=0,$$

the integral of which is $U=\varphi V$, art. 486. Since we have the equations $U=a$, and $V=b$, each of which is between three coordinates, we may consider them as the equations of two surfaces, and the coordinates being common, they must belong to the curve of intersection of the two surfaces. This being premised, since a and b are arbitrary constants, if in $U=a$ we give to x and y the values x' and y', we shall obtain for z a function of x', y', and a, which will determine a point of the surface of which $U=a$ is the equation. This point will change its position if we give successively different values to the arbitrary constant a; that is to say, by making a vary, the surface whose equation is $U=a$ will be made to pass through a new system of points. What has been said of $U=a$ will apply equally to $V=b$; we may conclude therefore that the curve of intersection of the two surfaces will continually change its position, and consequently describe a curve surface, in which a and b may be considered as two coordinates; and since the relation $a=\varphi b$, which connects these two coordinates, is arbitrary, we see that the determination of the function φ reduces itself to the problem of making a surface pass through a curve traced arbitrarily.

507. To show how this sort of problems may conduct to

analytical conditions, we will examine what the surface is, the equation to which is

$$y\frac{dz}{dx}=x\frac{dz}{dy} \quad . \; . \; . \quad (216).$$

We have seen, art. 477, that this equation has for its integral

$$z=\varphi(x^2+y^2) \quad . \; . \; . \quad (217),$$

and reciprocally we deduce from this integral

$$x^2+y^2=\Phi z\,;$$

if we cut the surface by a plane parallel to that of x, y, the section will have for its equation

$$x^2+y^2=\Phi c,$$

and representing the constant Φc by a^2, we shall have

$$x^2+y^2=a^2.$$

This equation belongs to a circle; and, consequently, the surface will possess this property, that every section made by a plane parallel to that of x, y, will be a circle.

508. This property is also indicated by the equation (216); for, by virtue of art. 26, we deduce from it

$$x=y\frac{dy}{dx},$$

which shows us that the subnormal is always equal to the abscissa, and this is a property of the circle.

509. The equation (217) indicating to us no other condition than that the sections parallel to the plane of x, y, must be circles, it follows that the law according to which the radii of these sections are to increase, is not comprised in the equation (217); and that, consequently, every surface of revolu-

tion will satisfy the problem; for we know that in surfaces of this sort the sections parallel to the plane of x, y, are always circles, and it is needless to say that the generating curve which, by its revolution, describes the surface, may be discontinuous, discontiguous, regular or irregular.

510. Let us investigate now the surface for which this generating curve should be a parabola AN (fig. 91), and suppose Fig. 91. that, on this hypothesis, the surface be cut by a plane AB, passing through the axis of z; the track of this plane along that of x, y, will be a straight line AL, which being drawn through the origin, will have for its equation $y=ax$; and if we represent by t the hypothenuse AQ of the right-angled triangle APQ, constructed along the plane of x, y, we shall have

$$t^2 = x^2 + y^2,$$

but t being the abscissa AQ of the parabola AN, of which QM $=z$ is the ordinate, we have, from the nature of the curve,

$$t^2 = bz,$$

putting for t^2 its value x^2+y^2, there results

$$z=\frac{1}{b}(x^2+y^2) \text{ or } z=\frac{1}{b}(a^2x^2+x^2)=\frac{x^2}{b}(a^2+1),$$

and making $\frac{1}{b}(a^2+1)=m$, we shall obtain

$$z=mx\ ;$$

so that the condition prescribed on the hypothesis of the generating curve being a parabola is that we should have

$$z=mx^2, \text{ when } y=ax.$$

511. We shall seek now to determine, by means of these conditions, the arbitrary function which enters into the equation (217). For this purpose, representing by U the quantity

x^2+y^2, which is affected by the sign φ, the equation (217) will become

$$z=\varphi U \quad . \; . \; . \; . \quad (218),$$

and we shall have the three equations

$$x^2+y^2=U, \; y=ax, \; z=mx^2.$$

Eliminating y by means of the two first of these, we shall obtain the value of x^2, which, being substituted in the third, will give us

$$z=m\frac{U}{a^2+1},$$

an equation which reduces itself to

$$z=\frac{1}{b}U,$$

since we have already supposed $\frac{1}{b}(a^2+1)=m$. This value of z being then substituted in the equation (218), it will be changed into

$$\varphi U=\frac{1}{b}U;$$

putting the value of U in this equation, we shall find

$$\varphi(x^2+y^2)=\frac{1}{b}(x^2+y^2),$$

and we see that the form of the function is determined. Substituting this value of $\varphi(x^2+y^2)$ in the equation (217), we shall have for the integral sought,

$$z=\frac{1}{b}(x^2+y^2),$$

an equation which possesses the property required, since the hypothesis of $y=ax$ gives us

$$z=mx^2.$$

512. This process is general; for suppose that the conditions on which the arbitrary constant is to be determined are that the integral give $F(x, y, z)=0$, when we have $f(x, y, z)=0$; we shall obtain a third equation by equating to U the quantity which is preceded by φ, and then by eliminating successively every two of the variables x, y, z, we shall obtain each of those variables in the function of U; putting these values in the integral, we shall arrive at an equation the first side of which will be φU, and the second side an expression also composed of U; replacing the value of U in terms of the variable, the arbitrary function will thus be determined.

Of the arbitrary functions which enter into the integrals of partial differentials of the second order.

513. The partial differential equations of the second order conduct to integrals which contain two arbitrary functions; the determination of these functions resolves itself into making a surface pass through two curves which may be discontinuous and discontiguous. To give an example, let us take the equation $\frac{d^2z}{dx^2}=0$, the integral of which, art. 491, is

$$z=x\varphi y+\psi y \quad . \; . \; . \quad (219).$$

Let Ax, Ay, Az, (fig. 92), be the coordinate axes; if we draw a plane KL parallel to that of x, z, the section of the surface, by this plane, will be a straight line; for, for all the points of this section, y being equal to Ap, if we represent Ap by a constant c, the quantities φy and ψy will become φc and ψc, and consequently may be replaced by two constants a and b, so that the equation (219) will become Fig. 92.

$$z=ax+b \quad . \; . \; . \; . \quad (220,$$

and will be that of the section made by the plane KL.

514. To determine the point in which this section meets the plane of y, z, making $x=0$, the equation (219) gives us

on this hypothesis $z = \psi y$, which indicates to us a curve *amb*, traced out along the plane of y, z. It would be easy to show, as in art. 502, that the section meets the curve *amb* in a point m; and since this section is a straight line, to determine its position, we have only to find a second point through which it must pass. For this purpose, we must observe, that when x is equal to 0, the equation (219) is reduced to

$$z = \psi y \,;$$

whilst, when x is equal to unity, the same equation is reduced to

$$z = \varphi y + \psi y \,,$$

and making, as before, $y = Ap = c$, these two values of z become

$$z = b, \quad z = a + b,$$

and determine two points m and r, taken in the same section mr, which we have seen to be a straight line. To construct these points, we must proceed in the following manner: we must trace arbitrarily along the plane of y, z, the curve *amb*, and through the point p, in which the secant plane KL meets the axis of y, raise the perpendicular $pm = b$, which will be an ordinate to the curve; we must then take, at the intersection HL of the secant plane with that of x, y, the part pp' equal to unity: through the point p' draw a plane parallel to that of y, z; and on this plane construct the curve $a'm'b'$, similar, and similarly disposed with the curve *amb*; then the ordinate $m'p'$ will be equal to mp; and if we produce $m'p'$ to r, the arbitrary quantity $m'r$ representing a, we shall determine the point r of the section.

If then we prolong, by the same process, all the ordinates of the curve $a'm'b'$, we shall construct a new curve $a'rb'$, which will be such, that, drawing through this curve and through *amb*, a plane parallel to that of x, z, the two points in which the curves are cut will belong to the same section of the surface.

515. It follows, from what has been said, that the surface may be constructed, by making the straight line *mr* move in such a manner that it shall always touch the two curves *amb*, *a rb'*.

516. This example will serve to show, in some slight degree, how the determination of the arbitrary functions, which complete the integrals of partial differential equations of the second order, resolves itself into making the surface pass through two curves, which, according to the arbitrary functions which serve to construct them, may be discontinuous, discontiguous, regular or irregular.

NOTES.

NOTE FIRST.

On the manner of finding the development of log (x + h).

The following is one of the processes employed for finding the logarithm of $x + h$. The development of log $(1 + x)$ is first investigated thus: we equate log $(1 + x)$ to a series of terms arranged according to the powers of x, observing that in this series there can be no term independent of x; for if we had

$$\log(1 + x) = A + Bx + Cx^2 + \&c.,$$

since this equation ought to hold good, whatever x be, it would follow that, on making $x = 0$, we should have $A = \log 1 = 0$; we shall therefore assume

$$\log(1 + x) = Ax + Bx^2 + Cx^3 + \&c. \quad . \quad . \quad . \quad (1);$$

changing x into z, we shall have similarly

$$\log(1 + z) = Az + Bz^2 + Cz^3 + \&c.;$$

and z being arbitrary, we may suppose that there exists between x and z the relation

$$(1 + x)^2 \text{ or } 1 + 2x + x^2 = 1 + z;$$

deducing from this equation the value of z, and substituting it in the equation (1), we shall find

$$\log(1+x)^2 = A(2x+x^2) + B(2x+x^2)^2 + C(2x+x^2)^3 + \&c.\,;$$

whence, developing and arranging according to the powers of x,

$$\log(1+x)^2 = 2Ax + \left.\begin{matrix} A \\ +4B \end{matrix}\right\} x^2 \left.\begin{matrix} +4B \\ +8C \end{matrix}\right\} x^3 \left.\begin{matrix} +\ \ B \\ +12C \\ +16D \end{matrix}\right\} x^4 + \&c. \left.\vphantom{\begin{matrix} a \\ b \\ c \end{matrix}}\right\} (2).$$

On the other hand, from the property of logarithms expressed in the equation $\log a^n = n \log a$, we have

$$\log(1+x)^2 = 2\log(1+x),$$

or, putting for $1+x$ its development (1),

$$\log(1+x)^2 = 2(Ax + Bx^2 + Cx^3 + \&c.)\,;$$

substituting this value of $\log(1+x)^2$ on the first side of the equation (2), we shall have an equation which must hold good, whatever x be; and consequently equating to each other the terms affected by the same powers of x, we shall obtain

$$2A = 2A,\quad A+4B = 2B,\quad 4B+8C = 2C,\quad \&c.\,;$$

whence we shall deduce

$$B = -\frac{A}{2},\quad C = -\frac{2B}{3} = \frac{A}{3},\quad \&c.,$$

and substituting these values, we shall find

$$\log(1+x) = A\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \&c.\right) + C.$$

When $x=0$, $\log 1 = 0 = C$, and therefore there is no constant to be added.

Making now $x = \frac{h}{x}$, and therefore

$$\log(1+x) = \log\left(1+\frac{h}{x}\right) = \log\frac{x+h}{x} = \log(x+h) - \log x,$$

we shall have

$$\log(x+h)-\log x=A\left(\frac{h}{x}-\frac{h^2}{2x^2}+\frac{h^3}{3x^3}-\frac{h^4}{4x^4}+\&c.\right) \quad . \quad (3),$$

and dividing by h,

$$\frac{\log(x+h)-\log x}{h}=A\left(\frac{1}{x}-\frac{h}{2x^2}+\frac{h^2}{3x^3}-\frac{h^3}{4x^4}+\&c.\right).$$

Passing to the limit, we shall find $\frac{d.\log x}{dx}=\frac{A}{x}$, and consequently the differential of $\log x$ is $\frac{Adx}{x}$. It will be seen that the constant A is no other than the modulus, and since in the Napierian system the modulus is supposed equal to unity, this hypothesis reduces the equation (3) to

$$\log(x+h)=\log x+\frac{h}{x}-\frac{h^2}{2x^2}+\frac{h^3}{3x^3}-\frac{h^4}{4x^4}+\&c$$

NOTE SECOND.

Considerations which prove the solidity of the principles of differentiation, and the means by which we may avoid employing Maclaurin's theorem in the differentiation of exponential quantities. A new process for arriving readily at the differentials of logarithmic and exponential quantities.

With the exception of the differentials of circular functions, which, as we have seen, are readily found by the formulæ of trigonometry, all the other monomial differentials, such, for example, as those of x^m, a^x, $\log x$, &c., have been deduced from the binomial theorem alone. We have, it is true, had recourse to the theorem of Maclaurin, in the determination of the constant A in the exponential formulæ, but we might have dispensed with it; for the value of A being determined from the equation 20 (art. 36), it is easily shown that the second side of that equation is no other than the development of the Napierian logarithm of a. For this purpose, if in the formula demonstrated by algebra, and also in the preceding note, viz.

$$\log(x+h)=\log x+\frac{h}{x}-\frac{h^2}{2x^2}+\frac{h^3}{3x^3}-\frac{h^4}{4x^4}+\&c.,$$

we make $x=1$ and $h=a-1$, we shall find

$$\log(1+a-1) \text{ or } \log a$$

$$=\log 1+\frac{a-1}{1}-\frac{(a-1)^2}{2}+\frac{(a-1)^3}{3}-\&c.\,.$$

and since $\log 1=0$,

$$\log a=\frac{a-1}{1}-\frac{(a-1)^2}{2}+\frac{(a-1)^3}{3}-\&c.\,;$$

which shows us that the second side of the equation (20) is equivalent to the Napierian logarithm of a, and that, consequently, we may change A into $\log a$, as was done art. 37.

It follows from this that the principles of differentiation rest all of them on the binomial theorem alone; and since that theorem has been demonstrated, in the elements of algebra, with all the rigour possible, we may conclude that our principles are founded on a firm basis.

We shall terminate this note with a new process for arriving at the differential of a , found (articles 36 and 37).

For this purpose, taking the equation $y=a^x$, in which the abscissa x is the logarithm of the ordinate y, if we assume any ordinates whatever y and z, we must have

$$y=a^{\log y},\quad z=a^{\log z}\quad .\quad .\ (1).$$

If now the abscissæ $\log y$ and $\log z$ of the ordinates y and z be increased by the same quantity h, and the corresponding ordinates be represented by y' and z', we shall obtain

$$y' = a^{h+\log y}, \quad z' = a^{h+\log z}:$$

and consequently, dividing these equations by the equations (1), we shall have

$$\frac{y'}{y} = a^h = \frac{z'}{z};$$

which will give us the proportion

$$y' : y :: z' : z,$$

and therefore

$$y' - y : y :: z' - z : z.$$

Transposing the means, and dividing the two first terms by h, we shall obtain from this

$$\frac{y'-y}{h} : \frac{z'-z}{h} :: y : z,$$

passing to the limit, this proportion will become

$$\frac{dy}{dx} : \frac{dz}{dx} :: y : z,$$

or

$$\frac{dx}{dz} : \frac{dx}{dy} :: y : z;$$

whence we deduce

$$z\frac{dx}{dz} = y\frac{dx}{dy}$$

From art. 69, we recognize, in these expressions, those of the subtangents at the points the ordinates of which are y and z, which shows us, therefore, that in the curve whose equation is $y = a^x$, the subtangent is constant. Representing its value by c, we shall have

$$y\frac{dx}{dy} = c,$$

whence

$$dx = c.\frac{dy}{y},$$

and consequently

$$d \log y = c\frac{dy}{y},$$

which agrees with art. 38; and supposing that the logarithms belong to the Napierian system, we shall have $c=1$, and consequently,

$$d.\log y = \frac{dy}{y}.$$

To deduce from this equation the differential of the exponential quantity a^x, making

$$a^x = z,$$

and taking the logarithms in the Napierian system, we shall have

$$\mathrm{L}a^x = \mathrm{L}z, \text{ or } x\mathrm{L}a = \mathrm{L}z;$$

differentiating, we shall find

$$dx\mathrm{L}a = \frac{dz}{z},$$

and consequently,

$$dz = zdx\mathrm{L}a, \text{ or } d.a^x = a^x dx\mathrm{L}a,$$

which is conformable to art. 37.

NOTE THIRD.

On the principle of the method of indeterminate coefficients.

It may be demonstrated that when an equation, such as

$$\mathrm{A}x^4 + \mathrm{B}x^3 + \mathrm{C}x^2 + \mathrm{D}x + \mathrm{E} = 0 \quad . \; . \; . \; . \quad (1),$$

holds good, whatever be the value of x, each of the coefficients A, B, C, D, E, must necessarily be $=0$.

For since x may have any value whatever, making $x=0$, the equation (1) will be reduced to $E=0$, and since E is independent of x, it will still be 0, though x have any other value than 0; the equation (1) is therefore reduced to

$$Ax^4+Bx^3+Cx^2+Dx=0,$$

whence, suppressing the common factor x, there results

$$Ax^3+Bx^2+Cx+D=0;$$

applying to this equation the same reasoning which we have employed in regard to the equation (1), we may prove that D is 0, and continuing the process, we shall find successively that the other coefficients are so also.

NOTE FOURTH.

On the integration of rational fractions, the denominators of which, being equated to 0, contain roots imaginary and equal.

The integration of rational fractions of this sort reducing itself to that of the formula $\int \frac{Mdz}{(\beta^2+z^2)^p}$, since the manner in which we have integrated this expression, art. 311, is a little complicated, we shall here point out another process which, though less direct, is often employed for arriving at the end in view.

We suppose

$$\int \frac{dz}{(\beta^2+z^2)^p}=\frac{Hz}{(\beta^2+z^2)^{p-1}}+K\int \frac{dz}{(\beta^2+z^2)^{p-1}} \quad \ldots \text{(1)}.$$

or, which comes to the same thing,

$$\int \frac{dz}{(\beta^2+z^2)^p} = Hz(\beta^2+z^2)^{1-p} + K\int \frac{dz}{(\beta^2+z^2)^{p-1}};$$

differentiating, we have

$$\frac{dz}{(z^2+\beta^2)^p} = Hdz(\beta^2+z^2)^{1-p} + H(1-p)(\beta^2+z^2)^{-p}2z^2dz \dots .$$
$$+ K\frac{dz}{(\beta^2+z^2)^{p-1}},$$

or,

$$\frac{dz}{(z^2+\beta^2)^p} = \frac{Hdz(\beta^2+z^2)}{(\beta^2+z^2)^p} + \frac{2H(1-p)z^2dz}{(\beta^2+z^2)^p} + K\frac{(\beta^2+z^2)dz}{(\beta^2+z^2)^p};$$

suppressing the common factors, we find

$$1 = H(\beta^2+z^2) + 2H(1-p)z^2 + K(\beta^2+z^2),$$

equating to each other the coefficients of z^2, as also those which are independent of z, we shall obtain

$$1 = H\beta^2 + K\beta^2, \quad H + 2(1-p)H + K = 0;$$

and these values give us

$$H = \frac{1}{2(p-1)\beta^2}, \quad K = \frac{2p-3}{2(p-1)\beta^2}:$$

H and K being thus known, if we substitute their values in the equation (1), we shall have

$$\int \frac{dz}{(\beta^2+z^2)^p} = \frac{1}{2(p-1)\beta^2} \cdot \frac{z}{(\beta^2+z^2)^{p-1}}$$
$$+ \frac{2p-3}{2(p-1)\beta^2} \int \frac{dz}{(\beta^2+z^2)^{p-1}} \quad \dots . \ (2);$$

and the integral of $\frac{dz}{(\beta^2+z^2)^p}$ will thus be made to depend on another, in which the index of the part within the brackets will be reduced by unity. If then in the formula (2) we assume $p=p-1$, we shall make the integral of $\frac{dz}{(\beta^2+z^2)^{p-1}}$ dependent on that of $\frac{dz}{(z^2+\beta^2)^{p-2}}$, and continuing thus to diminish the index of the part within the brackets by unity successively, we shall come at last to $\int\frac{dz}{\beta^2+z^2}$, the integral of which is . . $\frac{1}{\beta}\tan^{-1}.\frac{z}{\beta}$.

NOTE FIFTH.

On the development of the powers of sines and cosines in terms of the multiple arcs.

There is a formula of particular elegance, which gives the value of a power of the cosine in terms of the quantities $\cos x$, $\cos 2x$, $\cos 3x$, &c.; and a similar formula exists also for the sines: with these it is important to be acquainted; but before we turn our attention to them, we must proceed to give the demonstration of a remarkable imaginary formula, of which we shall often have to make use.

Suppose, then, that we have given the expression . . . $\cos^2\varphi+\sin^2\varphi$, which is the product of the two factors . . . . . $\cos\varphi+\sin\varphi\sqrt{-1}$, and $\cos\varphi-\sin\varphi\sqrt{-1}$; if we make . . . $\cos\varphi+\sin\varphi\sqrt{-1}=F\varphi$, we shall have, by differentiating,

$$\frac{dF\varphi}{d\varphi}=-\sin\varphi+\cos\varphi\sqrt{-1};$$

this equation being multiplied by $-\sqrt{-1}$, becomes

$$-\frac{dF\varphi}{d\varphi}\sqrt{-1}=\sin\varphi\sqrt{-1}+\cos\varphi,$$

and since by hypothesis the second side is equal to $F\varphi$, we have

$$-\frac{dF\varphi}{d\varphi}\sqrt{-1}=F\varphi,$$

whence we deduce

$$\frac{dF\varphi}{F\varphi}=-\frac{d\varphi}{\sqrt{-1}}=-\frac{d\varphi}{\sqrt{-1}}\times\frac{\sqrt{-1}}{\sqrt{-1}}=d\varphi\sqrt{-1};$$

integrating this, we find

$$\log F\varphi=\varphi\sqrt{-1}=(\varphi\sqrt{-1})\log e=\log e^{\varphi\sqrt{-1}};$$

passing to numbers, we have

$$F\varphi=e^{\varphi\sqrt{-1}},$$

and putting for $F\varphi$ its value, we obtain

$$\cos\varphi+\sin\varphi\sqrt{-1}=e^{\varphi\sqrt{-1}}\ \ldots\ldots\ (1).$$

This equation being true, whatever be the value of φ, we may change φ into $m\varphi$, and we shall have then

$$\cos m\varphi+\sin m\varphi\sqrt{-1}=e^{m\varphi\sqrt{-1}}.$$

There is another expression for this imaginary power of e, for the equation (1) being raised to the power m, gives us

$$(\cos\varphi+\sin\varphi\sqrt{-1})^m=e^{(\varphi\sqrt{-1})m}=e^{m\varphi\sqrt{-1}};$$

and the second sides of the two last equations being the same, we have, by equating the first sides,

$$(\cos\varphi+\sin\varphi\sqrt{-1})^m=\cos m\varphi+\sin m\varphi\sqrt{-1}\ \ldots\ldots\ (2).$$

If we make $\varphi=-\varphi$ in the equations (1) and (2), they will become

$$\cos(-\varphi)+\sin(-\varphi)\sqrt{-1}=e^{-\varphi\sqrt{-1}}\ \ldots\ldots\ (3),$$

$$(\cos(-\varphi)+\sin(-\varphi)\sqrt{-1})^m=\cos(-m\varphi)$$
$$+\sin(-m\varphi)\sqrt{-1}\ \ldots\ldots\ (4).$$

If now φ be represented by the arc AD (fig. 61), $-\varphi$ will be represented by AD′; and since these arcs have the same cosines, and the same sines with contrary signs, we shall have

$$\cos(-\varphi)=\cos\varphi,\ \sin(-\varphi)=-\sin\varphi.$$

We might prove, in the same manner, that

$$\cos(-m\varphi)=\cos m\varphi,\ \sin(-m\varphi)=-\sin m\varphi\ ;$$

and substituting these values in the equations (3) and (4), we shall obtain

$$\cos\varphi-\sin\varphi\sqrt{-1}=e^{-\varphi\sqrt{-1}}\ .\ .\ .\ (5),$$

$$(\cos\varphi-\sin\varphi\sqrt{-1})^m=\cos m\varphi-\sin m\varphi\sqrt{-1}\ .\ .\ .\ (6).$$

We will now investigate the development of $\cos^m x$ in terms of the multiple arcs, without employing the powers of the sine and cosine. For this purpose, assume

$$\cos x+\sin x\sqrt{-1}=u\ .\ .\ .\ (7),$$

$$\cos x-\sin x\sqrt{-1}=v\ .\ .\ .\ (8)\ ;$$

these equations being added, give

$$\cos x=\frac{1}{2}(u+v),$$

and consequently

$$\cos^m x=\frac{1}{2^m}(u+v)^m,\ \cos^m x=\frac{1}{2^m}(v+u)^m\ ;$$

developing these binomials by the usual formula, we obtain

$$\cos^m x=\frac{1}{2^m}\left(u^m+mu^{m-1}v+m\ .\ \frac{m-1}{2}u^{m-2}v^2+\&c.\right)$$

$$\cos^m x=\frac{1}{2^m}\left(v^m+mv^{m-1}u+m\ .\ \frac{m-1}{2}v^{m-2}u^2+\&c.\right)\ ;$$

and adding these equations, we find

$$2^{m+1}\cos^{m}x = u^{m}+v^{m}+muv(u^{m-2}+v^{m-2})$$
$$+m\,.\,\frac{m-1}{2}u^{2}v^{2}(u^{m-4}+v^{m-4})+\&c.\;.\;.\;.\;(9).$$

But from the formulæ (7) and (8) we deduce

$$u^{m}=(\cos x+\sqrt{-1}\sin x)^{m},\quad v^{m}=(\cos x-\sin x\sqrt{-1})^{m},$$

and putting on the second sides of these equations their values given by the formulæ (2) and (6), we have

$$\left.\begin{aligned} u^{m}&=\cos mx+\sin mx\sqrt{-1},\\ v^{m}&=\cos mx-\sin mx\sqrt{-1},\end{aligned}\right\}\;.\;.\;.\;(10);$$

whence

$$u^{m}+v^{m}=2\cos mx,\text{ and } u^{m}v^{m}=1,$$

and consequently

$$\begin{aligned} &.\; uv=1,\\ &u^{m-2}+v^{m-2}=2\cos(m-2)x,\; u^{m-2}v^{m-2}=1,\\ &u^{m-4}+v^{m-4}=2\cos(m-4)x,\; u^{m-4}v^{m-4}=1,\\ &\quad\&c.\quad=\quad\&c.\qquad\qquad\&c. \end{aligned}$$

Substituting these values in the equation (9), we shall find

$$\cos^{m}x=\frac{1}{2^{m+1}}[2\cos mx+2m\cos(m-2)x$$
$$+2m\,.\,\frac{(m-1)}{1.2}\cos(m-4)x+\&c.]\;.\;.\;.\;(11).$$

This development arising from that of $(u+v)^{m}$, will contain $(m+1)$ terms; if we make successively $m=2$, $m=3$, $m=4$, &c., and change the cosines of negative arcs into positive, by virtue of the equation $\cos-\varphi=\cos\varphi$, the following table will be formed:

$$\cos^2 x = \tfrac{1}{2}\cos 2x + \tfrac{1}{2},$$

$$\cos^3 x = \frac{\cos 3x}{4} + \frac{3\cos x}{4},$$

$$\cos^4 x = \frac{\cos 4x}{8} + \tfrac{1}{2}\cos 2x + \tfrac{3}{8}.$$

These calculations may be considerably abridged, for the terms equally distant from the extremities of the series are equal.

To prove this, it must be observed that the cosines which enter into the equation (11) are

$$\cos mx,\ \cos(m-2)x,\ \cos(m-4)x,\ \cos(m-6)x,\ \&c.,$$

or

$$\cos mx,\ \cos(m-2\times 1)x,\ \cos(m-2\times 2)x,\ \cos(m-2\times 3)x,\ \&c.,$$

in which series it will be seen that the number following the sign × in each term indicates the number of terms preceding it. Hence, the term, which has n terms before it, will be affected with $\cos(m-2n)x$; in respect to the term which has n after it, since the whole number of terms in the series is $m+1$, that which has n after it must hold the rank $m+1-n$, and, consequently, will have $m-n$ before it; it will therefore contain the expression

$$\cos[m-2(m-n)]x = \cos(-m+2n)x;$$

and since we have seen that we are at liberty to change the sign of the arc of which we have the cosine, we shall have

$$\cos(-m+2n)x = \cos(m-2n)x;$$

the terms equally distant from the two extremities of the series have therefore the same cosines, and since they have also the same coefficients *, the coefficients being those of the binomial

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* This may be seen by comparing the development of $(a+b)^m$ with that of $(b+a)^m$, written the opposite way.

theorem, it follows that those terms are equal. Thus, when m is odd, the number $m+1$ of the terms of the series will be even, and we shall have only to double the first $\frac{m+1}{2}$ terms, to obtain the value of the whole of the series: if m be even, $m+1$ will be odd, and then we must add to the middle term the double of the terms preceding it. This middle term will rank the $\left(\frac{m}{2}+1\right)$th in the series, and consequently will be affected by $\cos(m-m)=\cos 0=1$; it will therefore contain no cosine.

By a similar process, we may find the development of $\sin^m x$: for this purpose, subtracting the equation (8) from the equation (7), we get

$$2\sin x\sqrt{-1}=u-v, \text{ and therefore } \sin x=\frac{u-v}{2\sqrt{-1}};$$

raising the two sides of this equation to the power m, we shall have

$$\sin^m x=\frac{1}{(2\sqrt{-1})^m}(u-v)^m;$$

and if m be equal to an even number $2p$, we have

$$(u-v)^{2p}=[(u-v)^2]^p=[(v-u)^2]^p=(v-u)^{2p}$$

whence

$$(u-v)^m=(v-u)^m.$$

Developing now the equations

$$\sin^m x=\frac{1}{(\sqrt{2-1})^m}(u-v)^m, \text{ and } \sin^m x=\frac{1}{(2\sqrt{-1})^m}(v-u)^m,$$

and proceeding as we did above, we shall find

$$\sin^m x = \frac{1}{(2\sqrt{-1})^m}\left[\cos mx - m\cos(m-2)x + m\cdot\frac{m-1}{2}\cos(m-4)x - \&c.\right];$$

the imaginary quantity $2\sqrt{-1}$ being raised to an even power will disappear.

If m be equal to an odd number $2p+1$, we shall have

$$(u-v)^{2p+1} = (u-v)^{2p}\times(u-v) = (v-u)^{2p}\times-(v-u)$$
$$= -(v-u)^{2p+1},$$

whence

$$(u-v)^m = -(v-u)^m,$$

and

$$\sin^m x = \frac{(u-v)^m}{(2\sqrt{-1})^m},\ \sin^m x = -\frac{(v-u)^m}{(2\sqrt{-1})^m}\ \ldots\ (12);$$

developing $(u-v)^m$ and $(v-u)^m$ by the binomial theorem, and substituting these developments in the equations (12), added together, we shall have

$$2\sin^m x = \frac{1}{(2\sqrt{-1})^m}\left[u^m - v^m - m\cdot uv(u^{m-2} - v^{m-2}) + \&c.\right]\ \ldots\ (13).$$

Subtracting, then, the equations (10) from each other, multiplying the same equations together, and observing that the second operation gives us the sum of the squares of the sine and cosine of mx, which is equivalent to unity, we shall find

$$u^m - v^m = 2\sin mx\sqrt{-1},\ u^m v^m = 1;$$

and proceeding therefore in the same manner as before, we shall change the equation (13) into

$$\sin^m x = \frac{1}{2(2\sqrt{-1})^{m-1}}\left[\sin mx - m\sin(m-2)x + \frac{m\cdot(m-1)}{2}\sin(m-4)x\ \&c.\right]$$

Since, on this hypothesis, m is odd, the power $m-1$, to which the quantity $2\sqrt{-1}$ is raised, will be even, and the imaginary quantity will consequently disappear.

NOTE SIXTH.

On the method of determining the volumes of bodies the surface of which can be expressed by a function of a single variable.

When the solids are not those of revolution, we may sometimes determine the volume by means of a single integration, without making use of the formula (83) (art. 374). This we shall proceed to do in respect to the pyramid ABCD (fig. 66). For this purpose, let GFE be a section parallel to the base DBC; from the vertex A let fall the perpendicular AH on the base DBC, and express by x and h the parts AI and IH of this perpendicular, comprised between the point A and the planes DCB, GFE, respectively; then the area of the triangle GFE being diminished or increased according to the value which we give to x, it may be considered as a function of x, and we have therefore,

$$\text{GEF}=fx,\ \text{DBC}=f(x+h);$$

and the volume of the pyramid AGFE being thus made a function of x, we may suppose

$$\textit{volume}\ \text{AGEF}=\phi x,\ \textit{volume}\ \text{ADBC}=\phi(x+h).$$

Now it is evident that the truncated pyramid GB, which is the difference of these volumes, will be less than the volume of the prism, which has BCD for its base and h for its height, and will be greater than the volume of the prism, which has EFG for its base and h for its height; and the ratio of these prisms is

$$\frac{f(x+h)h}{fx.h}=\frac{f(x+h)}{fx};$$

since therefore, in the case of the limit, this ratio becomes equal to unity, still more will the ratio which exists between the truncated pyramid GB and one of the prisms be in that case equal to unity. Now the volume of the truncated pyramid is expressed by $\varphi(x+h)-\varphi x$, and the ratio of this volume to that of the prism, of which GFE is the base and h the height, will therefore be

$$\frac{\varphi(x+h)-\varphi x}{fx\cdot h}=\frac{\dfrac{d\varphi x}{dx}+\dfrac{d^2\varphi x}{dx^2}\cdot\dfrac{h}{2}+\&c.}{fx};$$

whence, passing to the limit, we shall have

$$\frac{d\varphi x}{fx.dx}=1, \text{ or } d\varphi x=fx.dx \quad . \quad . \quad . \quad . \quad (1).$$

We might have arrived at the same result by the method of infinitesimals; for, considering the pyramid as composed of an infinite number of slices parallel to the base, each slice might be supposed to be a prism, the base of which is fx, and dx the height, and $fx.dx$ would therefore be the element of the pyramid.

To determine the volume of the pyramid, let B be the area of its base and A its height; we shall have then

$$\mathrm{B} : fx :: \mathrm{A}^2 : x^2,$$

and therefore,

$$fx=\frac{\mathrm{B}x^2}{\mathrm{A}^2};$$

substituting this value in the equation (1), we shall find

$$d\varphi x=\frac{\mathrm{B}x^2}{\mathrm{A}^2}dx,$$

and, integrating,

$$\varphi x=\frac{\mathrm{B}x^3}{3\mathrm{A}^2}.$$

Since the volume AGEF, represented by φx, vanishes when

$x=0$, there is no constant to be added; and if we make $x=A$, we shall have, for the definite integral, the expression $\frac{BA}{3}$, which is that of the volume of the pyramid ACBD.

Generally, if the section GFE, instead of being a triangle, be any surface whatever, provided only that this section be a function of x, we may demonstrate, as we have already done for the pyramid, that the element of the solid has for its expression $fxdx$.

NOTE SEVENTH.

On the projection of a plane surface.

To demonstrate that the projection of a plane surface on another plane is equal to the product of that surface by the cosine of its inclination, let γ be the angle of inclination which a surface A makes with a surface B; since the surfaces are inclined to each other they must necessarily meet; let the axis of x be fixed in their common section, and suppose that the ordinates y of the surface A are drawn perpendicular to that axis; it is evident then that every ordinate y of that surface will have $y \cos\gamma$ for its projection on the other; and consequently the element of the surface A being represented by ydx, that of the surface B, or the projection of A on B, will be represented by $y \cos \gamma dx$; taking the integrals, we shall have

$$A = \int ydx, \quad B = \int y \cos \gamma dx = \cos \gamma \int ydx,$$

and eliminating $\int ydx$ between these equations, we shall find

$$B = A \cos\gamma.$$

NOTE EIGHTH.

The expression for the cosine of the angle formed by two planes derived directly by a new process.

Let it be proposed to resolve directly this problem: *to determine the cosine of inclination of two planes.*

Let DB, DC, CD, be the sections of a plane DBC (fig. 93) with the coordinate planes, the rectangular axes of which are taken along the lines AB, AC, AD; call AB, a; AC, b; AD, c; and represent by α, β, γ, the angles which the plane DBC makes with the planes of yz, xz, and zy; then the projections of the surface BCD on these planes being $\frac{bc}{2}, \frac{ac}{2}, \frac{ab}{2}$, respectively, we shall have, according to the preceding note,

$$\text{DBC}\cos\gamma=\frac{ab}{2},\ \text{DBC}\cos\alpha=\frac{bc}{2},\ \text{DBC}\cos\beta=\frac{ac}{2}\ldots(1);$$

each of these equations being squared, if we take their sum, replace the sum of the squares of the cosines by unity, and then extract the square root, we shall obtain

$$\text{DBC}=\tfrac{1}{2}\sqrt{a^2b^2+b^2c^2+a^2c^2};$$

and substituting this value in the first of the equations (1), we shall deduce from it

$$\cos\gamma=\frac{1}{\sqrt{1+\frac{c^2}{a^2}+\frac{c^2}{b^2}}}\ \ldots(2).$$

Let now $Ax+By+Cz+D=0$ be the equation of the plane DBC; if we make $y=0$, we shall have

$$Ax+Cz+D=0$$

for the equation of the section DB, which gives us

$$z = -\frac{A}{C}x - \frac{D}{C};$$

and since we know that in the equation of a straight line, put under this form, the coefficient of x represents the tangent of the angle which the straight line makes with the axis of x, we shall have

$$\tan DBA = -\frac{C}{A}.$$

But the right-angled triangle DBA gives us also

$$\tan DBA = \frac{c}{a};$$

whence, comparing these two values, we have

$$\frac{c^2}{a^2} = \frac{A^2}{C^2}.$$

Making then $x=0$, in the equation of the plane, to obtain that of the section with the plane of x, y, we shall find in like manner,

$$\frac{c^2}{b^2} = \frac{B^2}{C^2};$$

and substituting these values in the equation (2), there results

$$\cos\gamma = \frac{1}{\sqrt{1 + \frac{A^2}{C^2} + \frac{B^2}{C^2}}}.$$

If now we divide the equation of the plane by C, and differentiate it successively in respect to the variables x and y, we shall find,

$$\frac{dz}{dx} = -\frac{A}{C},\ \frac{dz}{dy} = -\frac{B}{C};$$

and substituting these values in that of $\cos\gamma$, we shall have, lastly,

$$\cos\gamma = \frac{1}{\sqrt{1+\left(\frac{dz}{dx}\right)^2+\left(\frac{dz}{dy}\right)^2}}$$

NOTE NINTH.

On the curve of double curvature, which may be constructed by means of two equations between three variables.

It is easy to prove that the equations (131), art. 427, belong to a curve of double curvature; for, changing y into z, and z into y, in order that the coordinate axis may be better adapted to our demonstration, we shall have the equations

$$z^2+2xy+y^3=0 \quad . \; . \; . \; . \quad (1),$$
$$2x+3y^2+2=0 \quad . \; . \; . \; . \quad (2).$$

If the first existed alone, we might, by means of it, construct a curve surface; for if, at all the points of the plane of x, y, which, as usual, we shall suppose to be horizontal, we erect perpendiculars, the values of the ordinates z will be determined by means of the equation (1), and we see that their extremities will constitute a curve surface. When any of those ordinates are imaginary, it is a mark that the surface does not extend through the points for which those imaginary ordinates exist.

If now we take into account the equation (2), we shall by it also establish a relation between x and y, which will require the feet of the ordinates z to be in the curve that belongs to the equation (2); in which case we see that the extremities of those ordinates will no longer form a surface, but a curve, and the system of the ordinates themselves will constitute a cylindrical surface, the intersection of which, with the plane of x, y, is given by the equation (2). It is the intersection of this surface with the one determined by the equation (1), which forms the curve of which we have been just speaking, and it is evidently one of double curvature, since we know that the intersection of two curve surfaces forms a curve of that nature.

NOTE TENTH.

On the value $\frac{0}{0}$, which, on the hypothesis of a particular solution, the eliminated constant sometimes assumes, when the equation of condition involves only variables.

The investigation of the particular solutions of a differential equation of the first order has conducted us (art. 439) to the case in which the equation of condition $q=0$ contains variables alone, and in which the combination of that equation with the complete integral produces the result $c=\frac{0}{0}$. This takes place when c enters only in the first degree into the complete integral $u=0$.

For that integral is then of the form

$$P+cQ=0 \quad . \; . \; . \quad (1),$$

where by P and Q we designate functions of x and y. This equation being differentiated in respect of x, y, and c, we shall have

$$dP+cdQ+Qdc=0 \quad . \; . \; . \quad (2);$$

and since the variables contained in P and Q are x and y, we may represent

$$dP \text{ by } Mdx+Ndy,$$
$$dQ \text{ by } mdx+ndy,$$

when, substituting these values in the equation (2), it will give us

$$dy=-\frac{(M+cm)}{N+cn}\,dx-\frac{Q}{N+cn}\,dc=0\ ;$$

and since, on the hypothesis of a particular solution, the term affected by *dc* vanishes, this gives us

$$\frac{Q}{N+cn}=0.$$

This equation, which cannot be reduced, since Q does not involve *c*, can only be satisfied by making $N+cn=\infty$, which gives $c=\infty$, or by making $Q=0$. The first supposition brings us to the case of a particular integral, since all integrals of that nature are comprised in the values which we give to *c* from zero up to infinity: to determine our particular solution therefore, if it exist, we have only the equation $Q=0$.

But when $Q=0$, the equation (2) is reduced to

$$dP+c\,dQ=0,$$

from which if we deduce the value of *c*, and substitute it in the equation (1), we shall obtain

$$P-\frac{dP}{dQ}Q=0\ ;$$

or, getting quit of the denominators,

$$P\,dQ-Q\,dP=0\quad .\ .\ .\ (3)\ ;$$

thus, in the present case, the complete integral $u=0$ and the proposed equation $U=0$ are no other than the equations (1) and (3). From the first we deduce

$$c=-\frac{P}{Q},$$

a value which is reduced to $\frac{0}{0}$, when P and Q have a common factor which is made to vanish by a value given to the variables.

This factor we will make appear by assuming $P=\lambda P'$ and $Q=\lambda Q'$; when the equations (1) and (3) will become

$$\lambda(P'+cQ')=0,\ \lambda(P'dQ-Q'dP)=0\ .\ .\ .\ (4).$$

The second of these, which represents the equation proposed, contains by hypothesis terms in dx and dy, which cannot be found but within the brackets, since λ, being a factor of the first of the equations (4), can contain only x and y; and since the operation of differentiating tends to diminish the indices of the variables, it follows, that the variables must be of a higher degree in the first equation than in the second, which is derived from it, and that, consequently, $P+cQ'$, which is not common to them, must be a function of x and y; and since also $P'+cQ'$ contains an arbitrary constant c, which is not found in λ, we see that $P'+cQ'$ has all the characteristics of the complete integral, and that λ, on the contrary, must be a factor unknown to the differential equation.

NOTE ELEVENTH.

Supplement to the theory of Lagrange on particular solutions, presented with certain modifications.—Method of obtaining the particular solution of a differential equation of the first order, without having recourse to the complete integral.—Demonstration of the property of particular solutions which causes the factor that renders a differential equation of the first order integrable to become infinite.

We have seen that a differential equation of the first order $Mdx+Ndy=0$ being given, we might consider it as the result of the elimination of a constant c between the complete integral and its differential $dy=pdx$, and that the result would be the same as if, supposing that constant variable, the elimination had been effected between the complete integral $F(x, y, c)=0$ and $dy=pdx+qdc$; on the condition however that we had $q=0$.

In like manner, if we assume that the differential equation of the second order,

$$M\frac{d^2y}{dx^2}+N\frac{dy}{dx}+P=0,$$

is the result of the elimination of a constant which has been made to vary, since we have in this case the two equations

$$dy=pdx+qdc,\ d.\frac{dy}{dx}=p'dx+q'dc\ .\ .\ .\ .\ .\ (1),$$

we see that, in order that they may be reduced to

$$dy=pdx,\ \text{and}\ d.\frac{dy}{dx}=p'dx,$$

we must have the two equations of condition

$$q=0,\ q'=0\,;$$

and that to establish these, it will not be sufficient to dispose of c alone, for that could fulfil only one condition; but since the integration of the equation of the second order has introduced two arbitrary constants into the complete integral, we must dispose of those two constants so that the equations $q=0$, $q'=0$ may be fulfilled; and it is needless to say that c will be one of those constants.

Similarly the determination of the particular solutions of a differential equation of the third order will depend on the equations $q=0$, $q'=0$, $q''=0$; and generally, to obtain a particular solution of the differential equation of the order n, we must have the number n of equations of condition:

$$q=0,\ q'=0,\ q''=0,\ q'''=0,\ \&c.\ .\ .\ .\ (2).$$

These may be put under another form; for the equations (1) show us that q and q' are no other than the multipliers of dc

in the differentials of y and $\frac{dy}{dx}$ taken in respect of x. We have therefore

$$q=\frac{dy}{dc},\ q'=\frac{d^2y}{dxdc};$$

and we see, generally, that the equations (2) reduce themselves to

$$\frac{dy}{dc}=0,\ \frac{d^2y}{dcdx}=0,\ \frac{d^3y}{dcdx^2}=0,\ \frac{d^4y}{dcdx^3}=0\ .\ .\ .\ (3).$$

It is essential to remark that these equations cannot be continued up to infinity: for $\frac{dy}{dc}$ being successively differentiated in respect to x in the expressions $\frac{dy}{dc}$, $\frac{d^2y}{dcdx}$, $\frac{d^3y}{dcdx^2}$, &c., we may consider $\frac{dy}{dc}$ as a certain function of x, which we will represent by Y; and supposing that x becomes $x+h$, Taylor's theorem will give us the development:

$$Y+\frac{dY}{dx}h+\frac{d^2Y}{dx^2}\frac{h^2}{1.2}+\frac{d^3Y}{dx^3}\frac{h^3}{1.2.3}+\&c.\ .\ .\ .\ (4),$$

or, restoring the value of Y,

$$\frac{dy}{dc}+\frac{d^2y}{dcdx}h+\frac{d^3y}{dcdx^2}\frac{h^2}{1.2}+\&c.;$$

and the coefficients of the powers of h being each of them 0 by virtue of the equations (3), which according to our hypothesis must hold good up to infinity, it would follow that when x became $x+h$, the equation (4) would be reduced to its first term Y, which shows that in this case Y, i. e. $\frac{dy}{dc}$, would be constant. But when $\frac{dy}{dc}$ is constant, c being combined only with constants, the equation $\frac{dy}{dx}=0$ must conduct us to $c=$ constant; and we see that the particular solution would then be changed into a particular integral, which we do not at all suppose.

It follows from the above that the equations (3) cannot hold good up to infinity; and on this consideration rests the solution of the following important problem, resolved by Lagrange, and which we shall give with certain modifications: *A differential equation of the first order being given, to find, without having recourse to the complete integral, the particular solution of which it may admit.*

Let u be the complete integral, which we suppose to be a function of x, y, and an arbitrary constant c; the differential of u will be represented by

$$mdx+ndy=0 \quad . \; . \; . \quad (5),$$

and may be put under the form,

$$dy=-\frac{m}{n}dx \quad . \; . \; . \quad (6).$$

In the case with which we are at present occupied, this equation is supposed to have retained the arbitrary constant [*]; and consequently we may eliminate this constant between $dy=-\frac{m}{n}dx$, and $u=0$. The value of c therefore being derived from the equation (5) in terms of x, y, and $\frac{dy}{dx}$, we shall obtain

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[*] If the complete integral should contain the arbitrary constant only in the first degree, and under the form ac, it would vanish by the differentiation, and the elimination of c would be impossible; but in this case q being constant, the equation proposed would not admit of a particular solution.

$$c=\varphi\left(x, y, \frac{dy}{dx}\right),$$

an equation which, for brevity's sake, we shall write thus

$$c=\varphi \quad . \quad . \quad . \quad (7),$$

and this value being substituted in the equation $u=0$, we shall have an equation of the first order, which we will designate by $U=0$, or rather by

$$M dx+N dy=0.$$

If, now, we differentiate $U=0$ in respect of the three variables x, y, and φ, we shall obtain

$$\frac{dU}{dx}dx+\frac{dU}{dy}dy+\frac{dU}{d\varphi}d\varphi=0 ;$$

and since y cannot vary except by reason of the arbitrary value which we give to x, this equation may be written thus:

$$\left(\frac{dU}{dx}+\frac{dU}{dy}\frac{dy}{dx}\right)dx+\frac{dU}{d\varphi}d\varphi=0 ; \quad . \quad . \quad . \quad (8).$$

But if we bear in mind that, in a function of two variables, the first term of the differential is obtained by considering one of those variables as constant, and the other as variable, we shall perceive that in the equation (8), which, under a certain point of view, contains only two variables, x and φ*, φ is constant in the term

$$\left(\frac{dU}{dx}+\frac{dU}{dy}\frac{dy}{dx}\right)dx ;$$

which term is no other than the differential of u taken in respect of the variables x, y, and in which the symbol φ ought to be substituted for c.

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* This results from y being treated as a function of x.

But this differential is given us by the equation (5), and since the second side of that equation indicates that the terms must all destroy each other on the first, independently of c, it must of course be the same when φ holds the place of the arbitrary constant c. It follows, therefore, that the part contained within the brackets in the equation (8) must be identically 0, and this equation consequently is reduced to

$$\frac{dU}{d\varphi}d\varphi = 0 \quad . \; . \; . \quad (9)\,;$$

which may be satisfied by making

$$d\varphi = 0 \text{ or } \frac{dU}{d\varphi} = 0 \quad . \; . \; . \quad (10),$$

and since it was only for brevity's sake that $\varphi\left(x, y, \frac{dy}{dx}\right)$ was replaced by φ, the first of the equations (10) is reduced to

$$d\varphi\left(x, y, \frac{dy}{dx}\right) = 0 \quad . \; . \quad (11),$$

a differential equation of the second order; which being integrated gives us

$$\varphi\left(x, y, \frac{dy}{dx}\right) = \text{constant} \quad . \; . \; . \quad (12).$$

On the other hand, the proposed equation $U = 0$ is between the same variables x, y, and $\frac{dy}{dx}$. We have, therefore, two equations of the first order corresponding to one and the same equation (11) of the second; and consequently by eliminating $\frac{dy}{dx}$ between them, we shall obtain a function of x and y and the arbitrary constant c contained in the equation (12); and the result of this operation will therefore be the complete integral (art. 429). To effect the elimination required, we have only to clear the equation $U=0$ of ϕ by means of the equation $\varphi=constant$; for then all the terms in $\frac{dy}{dx}$ contained in φ, and which do not otherwise exist, will disappear. This evidently comes to the same thing with changing φ into c in the equation $U=0$, which brings us back to $u=0$.

If the elimination of $\frac{dy}{dx}$ between the integral of the second factor of the equation (9) and the proposed one $U=0$ bring us back to the complete integral, it will be seen also that the elimination of $\frac{dy}{dx}$ between $U=0$ and the other factor of the equation (9) will conduct us to the particular solution. For if we eliminate $\frac{dy}{dx}$ between $U=0$, and $\frac{dU}{d\varphi}=0$, we see at once that we shall not introduce any arbitrary constant into the result, as in the preceding operation, since here the elimination is effected without first integrating $\frac{dU}{d\varphi}$; and it follows, therefore, that the elimination of $\frac{dy}{dx}$ between these two differential equations of the first order cannot conduct us to the complete integral, which must necessarily contain an arbitrary constant. To proceed to this elimination, we must observe that it is reduced to the eliminating of φ; since $\frac{dy}{dx}$ being found nowhere but in φ, will disappear from the result along with that expression; and since this result retains no trace of c, we see that this comes to the same thing with eliminating c between $u=0$ and $\frac{du}{dc}=0$, which are what $U=0$ and . . . $\frac{dU}{d\varphi}=0$ become when φ is changed into c. But $\frac{du}{dc}$ being the differential coefficient of dc, we see that this elimination of c between u and $\frac{du}{dc}=0$ is precisely the operation which was gone through for the arriving at a particular solution.

We shall inquire now how we are to satisfy the condition expressed by the second of the equations (10); and for this purpose, if we replace φ by its value given by the equation (7), we shall obtain

$$\frac{dU}{dc}=0 \quad . \; . \; . \; . \quad (13).$$

We do not see at first how we can effect this differentiation in respect of c, which having been eliminated from U, ought not to be found in dU; but it must be observed that this elimination of c intimates only that U is a function of x, y, and $\frac{dy}{dx}$, and that consequently dU can be of no other form than

$$Pdx+Qdy+Rd.\frac{dy}{dx}=0 \quad . \; . \; . \; . \quad (14);$$

and though c do not explicitly appear in this value of dU, it must at least be found in it implicitly; for we know that y is a function of x and c, and consequently dy and $d.\frac{dy}{dx}$ must be of the following forms:

$$\left.\begin{aligned} dy&=\frac{dy}{dx}dx+\frac{dy}{dc}dc \\ d.\frac{dy}{dx}&=\frac{d^2y}{dx^2}dx+\frac{d^2y}{dxdc}dc \end{aligned}\right\} \quad . \; . \; . \; . \quad (15).$$

On the hypothesis of c being constant, these values are reduced to

$$dy=\frac{dy}{dx}dx,$$

$$d.\frac{dy}{dx}=\frac{d^2y}{dx^2}dx,$$

equations, the second sides of which express the condition that the differentiation be made in respect of x alone, a condition which we tacitly admit in the equation (14), when we suppose c constant in it; but when c is variable, we must put in the equation (14) the values of dy and $d.\frac{dy}{dx}$ given by the equations (15), and we shall have

$$\mathrm{P}dx+\mathrm{Q}\left(\frac{dy}{dx}dx+\frac{dy}{dc}dc\right)+\mathrm{R}\left(\frac{d^2y}{dx^2}dx+\frac{d^2y}{dxdc}dc\right)=0 \ldots (16).$$

This then is what $d\mathrm{U}$ becomes when c is considered as variable, and we see that we have

$$\frac{d\mathrm{U}}{dc}=\mathrm{Q}\frac{dy}{dc}=\mathrm{R}\frac{d^2y}{dxdc} \ \ldots\ldots (17).$$

If now we pass to the hypothesis of a particular solution, we have, by virtue of the equation (13), $\frac{d\mathrm{U}}{dc}=0$, which reduces the preceding equation to

$$\mathrm{Q}\frac{dy}{dc}+\mathrm{R}\frac{d^2y}{dxdc}=0 \ldots\ldots (18);$$

and if we suppose that this equation does not contain any transcendental quantities, and we have taken care, in the subsequent operations, to get rid of surds by raising to different powers, and also of fractions, the terms P and Q which the equation (14) contains cannot become infinite. This being premised, since $\frac{dy}{dc}$ is 0, by virtue of the equation (143) (art. 437), which expresses the condition of a particular solution being possible, we see that the equation (18) is reduced to

$$\mathrm{R}\frac{d^2y}{dx^2}=0 \ldots\ldots (19).$$

There may be two cases; $\frac{d^2y}{dx^2}$ may be 0, or it may not; on the second hypothesis it will be the factor R which, becoming 0, satisfies the equation (19); but if, on the contrary, $\frac{d^2y}{dx^2}$ be 0, the equation (18) is satisfied independently of Q and R; and consequently Q and R may be of finite values. It must not however be concluded from this that R is not 0; for if, treating y as a function of x, we differentiate the equation (18) in respect of that independent variable x, we find

$$R\frac{d^3y}{dx^2dc}+\frac{d^2y}{dxdc}\left(Q+\frac{dR}{dx}\right)+\frac{dQ}{dx}\frac{dy}{dc}=0\,^{*} \quad . \; . \; . \; . \quad (20);$$

and since the quantities $\frac{d^2y}{dxdc}$ and $\frac{dy}{dx}$ are each 0, and their coefficients cannot become infinite according to the remark made in respect to the equation (18), it follows that the equation (20) is reduced to $R\frac{d^3y}{dx^2dc}=0$, and consequently gives $R=0$, when $\frac{d^3y}{dx^2dc}$ is not so. If, however, $\frac{d^3y}{dx^2dc}$ should happen to be 0, it might be proved in the same manner that . . . $R\frac{d^4y}{dx^3dc}=0$, and that $\frac{d^4y}{dx^3dc}$ must be 0 in order that R may not be so. Continuing the same reasoning, we come at last to a differential coefficient $\frac{d^ny}{dx^{n-1}dc}$ which will not be 0, since it has been demonstrated that the equations (3) cannot be continued up to infinity; and it follows, therefore, from this demonstration, that R, which always retains the same value, being 0 in one case, must be so in all. But since R is 0, the equation

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[*] It must be observed that what we represent in a brief manner by . . . . $\frac{dR}{dx}dx$ and $\frac{dQ}{dx}dx$ is in fact

$$\left(\frac{dR}{dx}+\frac{dR}{dy}\frac{dy}{dx}\right)dx \text{ and } \left(\frac{dQ}{dx}+\frac{dQ}{dy}\frac{dy}{dx}\right)dx.$$

(14), put under the form

$$P+Q\frac{dy}{dx}+R\frac{d^2y}{dx^2}=0 \quad . \; . \; . \; . \quad (21),$$

is reduced to

$$P+Q\frac{dy}{dx}=0 \quad . \; . \; . \; . \quad (22).$$

On the other hand, the same equation (21) gives us

$$\frac{d^2y}{dx^2}=-\frac{P+Q\frac{dy}{dx}}{R},$$

and we see that, on our hypothesis of a particular solution, the equation (22) and the value of R, which is 0, reduce this value of $\frac{d^2y}{dx^2}$ to $\frac{0}{0}$.

It follows from this theory that, in the case in which a particular solution may exist, we have the equation $\frac{dU}{dc}=0$*, and that this equation requires, as a necessary consequence, that the value of $\frac{d^2y}{dx^2}$ reduce itself to $\frac{0}{0}$. The two terms of this fraction, i. e. the numerator and denominator of the fraction which expresses the value of $\frac{d^2y}{dx^2}$, being each equated to zero, will furnish us with two equations which, if they agree with $U=0$, will give the particular solution.

Let us take, for example, the equation

$$x+y\frac{dy}{dx}=\frac{dy}{dx}\sqrt{x^2+y^2-c^2} \quad . \; . \; . \quad (23).$$

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* We have seen that the equation $U=0$ was no other than the proposed one, considered as the result of the elimination of c; as to the one $\frac{dU}{dc}=0$, it intimates merely that the terms which, in the proposed equation, arise from the variation of the arbitrary constant are 0.

Squaring this, to get quit of the surd, and reducing, we shall have

$$x^2+2xy\frac{dy}{dx}+\frac{dy^2}{dx^2}(c^2-x^2)=0;$$

differentiating, then, considering dx as constant, we obtain

$$\frac{d^2y}{dx^2}=\frac{xdx+ydy}{(x^2-c^2)dy-xydx};$$

and equating the two terms of this fraction to zero, and dividing by dx, we shall have

$$x+y\frac{dy}{dx}=0,\ (x^2-c^2)\frac{dy}{dx}-xy=0\ \ .\ .\ .\ (24);$$

eliminating $\frac{dy}{dx}$ between these equations, and then suppressing the common factor, we shall find

$$y^2+x^2-c^2=0;$$

and since this equation satisfies the one proposed, we see that it is a particular solution.

Let us see now whether the equation

$$y-x\frac{dy}{dx}=x\sqrt{1+\frac{dy^2}{dx^2}}$$

admits of a particular solution. For this purpose, getting quit of the surd by squaring the two sides, and reducing, we shall find

$$y^2-2xy\frac{dy}{dx}-x^2=0;$$

and differentiating, there will result

$$\frac{d^2y}{dx^2}=-\frac{x\left(\frac{dy^2}{dx^2}+1\right)}{xy};$$

an equation which is reduced to $\frac{0}{0}$ when $x=0$; but this hypothesis not satisfying the one proposed, it cannot conduct us to a particular solution.

Conformably to the observation which was furnished us by the equations (149) and (150) (art. 442), it will not be necessary to limit ourselves to the hypothesis of y being a function of x; but supposing, then, x to be a function of y, we may investigate the particular solutions which may be given by equating the value of $\frac{d^2x}{dy^2}$ to $\frac{0}{0}$.

A particular solution producing the mutual destruction of the terms of the differential equation to which it belongs, it is no other than a factor which may be made to appear by means of a transformation. We have seen, for instance, that . . . . $x^2+y^2-a^2=0$ was the particular solution of the equation

$$(xdx+ydy)^2=dy^2(x^2+y^2-a^2) \quad . . . (25);$$

if we make $x^2+y^2-a^2=z^2$, we shall have

$$xdx+ydy=zdz;$$

and substituting these values in the equation (25), it will become

$$z^2(dz^2-dy^2)=0:$$

which shows that in reality the particular solution represented by z^2 is a common factor of the equation proposed. Another property of particular solutions is that of causing the factor, which renders the proposed differential complete, to become infinite. To demonstrate this, we shall put the complete integral under the form $u=constant$. Any value which satisfies this equation must therefore give $du=0$, since the differential of a constant is 0; and reciprocally any value which does not satisfy the equation $u=constant$ cannot give $du=0$. But this last case is precisely that of a particular solution which, since it does not satisfy the complete integral, cannot produce the mutual destruction of the terms which compose its immediate differential; and this immediate differential is no other than $\lambda(Mdx+Ndy)$. It is necessary, therefore, that the particular solution should not render equal to zero the second side of the equation

$$\lambda(\mathrm{M}dx+\mathrm{N}dy)=du,$$

or, which comes to the same thing, of the one

$$\lambda\left(\mathrm{M}+\mathrm{N}\frac{dy}{dx}\right)=\frac{d(u)}{dx}{}^{*};$$

from which we deduce

$$\lambda=\frac{\frac{d(u)}{dx}}{\mathrm{M}+\mathrm{N}\frac{dy}{dx}}\quad .\ .\ .\ (26).$$

But, from its nature, the particular solution, though it do not satisfy the equation $\lambda\left(\mathrm{M}+\mathrm{N}\frac{dy}{dx}\right)=0$, does, however, satisfy the one $\mathrm{M}+\mathrm{N}\frac{dy}{dx}=0$, i. e. it renders the first side 0; and this, therefore, reduces the equation (26) to

$$\lambda=\frac{\frac{d(u)}{dx}}{0}=\infty.$$

For example, the equation

$$xdx+ydy=dy\sqrt{x^2+y^2-a^2}\quad .\ .\ .\ .\ (27)$$

becomes a complete differential when we multiply it by

$$\frac{1}{\sqrt{x^2+y^2-a^2}},$$

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[*] We designate thus the total differential of u taken on the supposition of x being the independent variable, in order that $\frac{d(u)}{dx}$ may not be confounded with the differential coefficient $\frac{du}{dx}$, which supposes that all the other variables except x are considered as constant in that term.

and gives

$$\frac{xdx+ydy}{\sqrt{x^2+y^2-a^2}}=dy\,;$$

and we see that the particular solution $x^2+y^2-a^2=0$ renders the factor

$$\frac{1}{\sqrt{x^2+y^2-a^2}}=\infty\,.$$

NOTE TWELFTH.

A new demonstration in respect to the integration of partial differential equations.

We have seen, art. 486, that if in integrating the equation

$$\frac{dz}{dx}+\mathrm{M}\frac{dz}{dy}+\mathrm{N}=0\ \ .\ .\ .\ (1),$$

where M and N are functions of x, y, and z, we obtain two integrals $\mathrm{U}=a$ and $\mathrm{V}=b$, we must have necessarily $a=\varphi b$. The demonstration of this theorem being highly important, we have endeavoured to give to it the last degree of rigour in the following manner: U and V being functions of x, y, and z, the constants a and b may also be considered as functions of the same variables, by virtue of the equations $\mathrm{U}=a$ and $\mathrm{V}=b$; if therefore we differentiate these equations successively, we shall have

$$\left.\begin{aligned} da&=\mathrm{X}dx+\mathrm{Y}dy+\mathrm{Z}dz\\ db&=\mathrm{X}'dx+\mathrm{Y}'dy+\mathrm{Z}'dz\end{aligned}\right\}\ .\ .\ .\ (2);$$

and since these differentials ought to be each 0, by reason of a and b being constants, the equations $da=0$, $db=0$, give us the following ones:

$$\left.\begin{aligned} \mathrm{X}dx+\mathrm{Y}dy+\mathrm{Z}dz&=0\\ \mathrm{X}'dx+\mathrm{Y}'dy+\mathrm{Z}'dz&=0\end{aligned}\right\}\ .\ .\ .\ (3).$$

If in these equations, divided by dx, we substitute the values of dz and dy, deduced from the equations

$$dz+N dx=0,\ dy-M dx=0\ .\ .\ .\ (24),$$

given art. 478, we shall have

$$X+YM-ZN=0,\ X'+Y'M-Z'N=0.$$

From these equations we deduce

$$M=\frac{ZX'-XZ}{Z'Y-ZY'},\ N=\frac{X'Y-Y'X}{Z'Y-ZY'};$$

and substituting these values of M and N in the equation (1), we shall obtain

$$\frac{dz}{dx}+\frac{ZX'-XZ'}{Z'Y-ZY'}\frac{dz}{dy}+\frac{X'Y-Y'X}{Z'Y-ZY'}=0\ .\ .\ .\ (5);$$

the coefficients $\frac{dz}{dx}$ and $\frac{dz}{dy}$ are deduced from the equations (4), which give

$$\frac{dz}{dx}=-N,\ \frac{dy}{dx}=M,\ \frac{dz}{dy}=\frac{dz}{dx}\frac{dx}{dy}=-\frac{N}{M}\ .\ .\ .\ (6);$$

and substituting these values of $\frac{dz}{dx}$ and $\frac{dz}{dy}$ in the equation (5), and getting quit of the denominator, we shall find

$$-(Z'Y-ZY')N-(ZX'-XZ)\frac{N}{M}+X'Y-XY'=0\ .\ .\ (7).$$

The quantities X, Y, Z, which enter into this equation, are not always known, since they are not given except by differentiating the equations $U=a$, and $V=b$; we must therefore proceed to eliminate X, Y, Z, from our result. For this purpose, considering z as a function of x and y, we shall deduce from the equations (2),

$$\frac{da}{dx}=X+Z\frac{dz}{dx},\quad \frac{db}{dx}=X'+Z'\frac{dz}{dx},$$

$$\frac{da}{dy}=Y+Z\frac{dz}{dy},\quad \frac{db}{dy}=Y'+Z'\frac{dz}{dy};$$

substituting in these expressions the values of $\frac{dz}{dx}$ and $\frac{dz}{dy}$, given by the equations (6), and deducing the values of X, Y, X′ and Y′, we shall find

$$X=\frac{da}{dx}+ZN, \quad X'=\frac{db}{dx}+Z'N,$$

$$Y=\frac{da}{dy}+\frac{ZN}{M}, \quad Y'=\frac{db}{dy}+\frac{Z'N}{M};$$

and putting these values of X, Y, X′, and Y′, in the equation (7), and reducing, we shall obtain

$$\frac{da}{dx}\frac{db}{dy}=\frac{da}{dy}\frac{db}{dx} \quad . \; . \; . \; . \quad (8).$$

This equation shows us that a is a function of b; and, in fact, if we have $a = Fb$, by differentiating this equation, we shall find $da=\phi b\, db$, whence we shall deduce

$$\frac{da}{dx}=\phi b\frac{db}{dx}, \quad \frac{da}{dy}=\varphi b\frac{db}{dy};$$

and eliminating φb, we shall obtain the equation (8).

THE END.

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